

On generic chaining and the smallest singular value of random matrices with heavy tails

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Abstract

We present a very general chaining method which allows one to control the supremum of the empirical process $\sup_{h \in H} |N^{-1} \sum_{i=1}^N h^2(X_i) - \mathbb{E}h^2|$ in rather general situations. We use this method to establish two main results. First, a quantitative (non asymptotic) version of the classical Bai-Yin Theorem on the singular values of a random matrix with i.i.d entries that have heavy tails, and second, a sharp estimate on the quadratic empirical process when $H = \{\langle t, \cdot \rangle : t \in T\}$, $T \subset \mathbb{R}^n$ and μ is an isotropic, unconditional, log-concave measure.

1 Introduction

The main goal of this article is to obtain a non-asymptotic version of the Bai-Yin Theorem [5] on the largest and smallest singular values of certain random matrices. The Bai-Yin theorem asserts the following:

Theorem 1.1 *Let $A = A_{N,n}$ be an $N \times n$ random matrix with independent entries, distributed according to a random variable ξ , for which*

$$\mathbb{E}\xi = 0, \quad \mathbb{E}\xi^2 = 1 \quad \mathbb{E}\xi^4 < \infty.$$

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If $N, n \rightarrow \infty$ and the aspect ratio n/N converges to $\beta \in (0, 1]$, then

$$\frac{1}{\sqrt{N}} s_{\min}(A) \rightarrow 1 - \sqrt{\beta}, \quad \frac{1}{\sqrt{N}} s_{\max}(A) \rightarrow 1 + \sqrt{\beta},$$

almost surely, where s_{\max} and s_{\min} denote the largest and smallest singular value of A .

Also, without the fourth moment assumption, $s_{\max}(A)/\sqrt{N}$ is almost surely unbounded.

The main result of this article is a quantitative version of the Bai-Yin Theorem.

Quantitative Bai-Yin Theorem. For every $q > 4$ and $L > 0$, there exist constants c_1, c_2, c_3 and c_4 that depend only on q and L for which the following holds. For every integer n , $\beta \in (0, 1]$ and $N = n/\beta$, let $A = A_{N,n} = (\xi_{i,j})$ be an $N \times n$ random matrix with independent, symmetric entries, distributed according to a random variable ξ , satisfying $\mathbb{E}\xi^2 = 1$ and $\mathbb{E}|\xi|^q \leq L$. Then, for any $n \geq c_1$, with probability at least $1 - c_2/(\beta n^{c_3})$,

$$1 - c_4\sqrt{\beta} \leq \frac{1}{\sqrt{N}} s_{\min}(A) \leq \frac{1}{\sqrt{N}} s_{\max}(A) \leq 1 + c_4\sqrt{\beta}.$$

The proof of this result is based on the analysis of a more general scenario which has been studied extensively in recent years, in which the given matrix has independent rows, selected according to a reasonable measure on \mathbb{R}^n , rather than a matrix with i.i.d. entries; and unlike the classical random matrix theory approach, one is naturally interested in the non-asymptotic behavior of the largest and smallest singular values of $\Gamma = N^{-1/2} \sum_{i=1}^N \langle X_i, \cdot \rangle e_i$ as a function of N and n . We refer the reader to the surveys [34, 28] and references therein for the history and recent developments in the non-asymptotic theory of random matrices.

We will focus on the following questions:

Question 1.2 Let μ be a symmetric measure on \mathbb{R}^n and let $(X_i)_{i=1}^N$ be selected independently according to μ .

1. Let $\Sigma_N = \frac{1}{N} \sum_{i=1}^N X_i \otimes X_i$ be the sample covariance matrix and $\Sigma = \mathbb{E}(X \otimes X)$. Given $\varepsilon > 0$, is it true that with high probability, if $N \geq c(\varepsilon)n$ then $\|\Sigma_N - \Sigma\|_{2 \rightarrow 2} \leq \varepsilon$?

2. If X is an isotropic vector (that is, $\mathbb{E}\langle X, x \rangle^2 = \|x\|_{\ell_2^n}^2$ for every $x \in \mathbb{R}^n$), are there “canonical” high probability bounds on $s_{\max}(\Gamma)$ and $s_{\min}(\Gamma)$? For example, under what conditions on μ are $s_{\max}(\Gamma)$ and $s_{\min}(\Gamma)$ of the order of $1 \pm c\sqrt{n/N}$ – like in the Bai-Yin Theorem?

Observe that the two questions are very similar. For example, it is straightforward to verify that if μ is isotropic, then both parts can be resolved by estimating the supremum of the empirical process

$$\sup_{t \in S^{n-1}} \left| \frac{1}{N} \sum_{i=1}^N \langle X_i, t \rangle^2 - \mathbb{E} \langle X, t \rangle^2 \right|. \quad (1.1)$$

And, in view of the second part of Question 1.2, we will be especially interested in the case $N \sim n$, that is, while keeping the aspect ratio n/N constant.

When studying measures on \mathbb{R}^n in this context, it is natural to divide the assumptions into two types: one on the ℓ_p^n norm of X and the other on moments of linear functionals $\langle x, \cdot \rangle$.

To formulate the moment assumption we will use here, recall that for $\alpha \geq 1$, the ψ_α Orlicz norm of random variable Z is defined by

$$\|Z\|_{\psi_\alpha} = \inf \{c > 0 : \mathbb{E} \exp(|Z|^\alpha / c^\alpha) \leq 2\},$$

and there are obvious extensions for $0 < \alpha < 1$. It is standard to verify that for every $\alpha > 0$, $\|Z\|_{\psi_\alpha}$ is equivalent to $\sup_{q \geq 1} \|Z\|_{L_q} / q^{1/\alpha}$.

Assumption 1.3 *For $p, q \geq 2$, a symmetric measure μ satisfies a p -small diameter, L_q moment assumption with constants κ_1 and κ_2 , if a random vector X distributed according to μ satisfies that*

$$\|X\|_{\ell_p^n} \leq \kappa_1 n^{1/p} \text{ a.s., and for every } x \in S^{n-1}, \|\langle x, \cdot \rangle\|_{L_q} \leq \kappa_2. \quad (1.2)$$

μ satisfies a small diameter ψ_α moment assumption if the ψ_α norm replaces the L_q one in (1.2).

One should note that with very few exceptions, both parts of Assumption 1.3 are needed if one wishes to address Question 1.2.

The p -small diameter component, i.e. that $\|X\|_{\ell_p^n} \leq \kappa_1 n^{1/p}$ almost surely, is rather standard. Although it does not hold as stated even for a vector with i.i.d. gaussian entries, one may assume it without loss of generality unless N is much larger than n . Indeed, in typical situations $Pr(\|X\|_{\ell_p^n} \geq tn^{1/p})$ decays very quickly both in t and in n . Therefore, $\max_{i \leq N} \|X_i\|_{\ell_p^n} / n^{1/p}$ is bounded with very high probability, unless N is considerably larger than n (see Section 2 for more details). Hence, if $N \sim n$, which is the range we shall be interested in, a conditioning argument allows one to make the p -small diameter assumption.

Question 1.2 has been studied under the 2-small diameter assumption. In [27], Rudelson showed that if $\|X\|_{\ell_2^n} \leq \kappa_1 \sqrt{n}$ almost surely then for every $N \geq c_1 n \log n$, with probability at least 0.99,

$$1 - c_2 \sqrt{\frac{n \log n}{N}} \leq s_{\min}(\Gamma) \leq s_{\max}(\Gamma) \leq 1 + c_2 \sqrt{\frac{n \log n}{N}}, \quad (1.3)$$

and c_1, c_2 are constants that depend only on κ_1 .

It is straightforward to verify that this bound is optimal by considering the uniform measure on the set of coordinate vectors $\{\sqrt{n}e_1, \dots, \sqrt{n}e_n\}$, which results in the coupon-collector problem. Thus, given $\varepsilon > 0$, one requires at least $c(\varepsilon)n \log n$ random points to ensure that the sample covariance matrix ε -approximates the true covariance. Of course, [27] does not lead to a nontrivial estimate in the second part of Question 1.2, i.e. if the aspect ratio $n/N \rightarrow \beta \in (0, 1]$ and $n \rightarrow \infty$, and in particular, (1.3) can not yield a Bai-Yin type of bound. Any hope of getting the desired bounds in Question (1.2) requires additional assumptions on X .

Turning to the moments component of Assumption 1.3, note that a bound on the L_q moments of linear functionals means that $\|\langle x, \cdot \rangle\|_{L_q} \lesssim \|x\|_{\ell_2^n}$, and if, in addition, X is isotropic, the norms are equivalent. Moreover, in a similar fashion, a ψ_α assumption combined with isotropicity implies that the ψ_α and ℓ_2^n norms are equivalent.

Consider a situation when one only assumes such a moment condition. It is standard to verify that under a ψ_2 assumption, in which linear functionals exhibit a κ_2 -subgaussian tail behavior (i.e., $Pr(|\langle X, x \rangle| \geq t\kappa_2\|x\|_{\ell_2^n}) \leq 2\exp(-t^2/2)$), then with probability at least $1 - 2\exp(-c_3n)$,

$$s_{\min}(\Gamma), s_{\max}(\Gamma) \in [1 - c_4\sqrt{n/N}, 1 + c_4\sqrt{n/N}]$$

Indeed, a Bernstein type inequality shows that for each $x \in S^{n-1}$ and $0 < t < 1/\kappa_2$, $Pr(|N^{-1} \sum_{i=1}^N \langle X_i, x \rangle^2 - \mathbb{E} \langle X, x \rangle^2| \geq t) \leq 2\exp(-c_5 N t^2)$. And, if one is to obtain an estimate on the empirical process (1.1), one has to control a $1/2$ net on the sphere, which is of cardinality $\sim \exp(c_6 n)$. The tradeoff between the complexity of the indexing set and the concentration at hand shows that with the desired probability, $\sup_{t \in S^{n-1}} |N^{-1} \sum_{i=1}^N \langle X_i, t \rangle^2 - \mathbb{E} \langle X, x \rangle^2| \lesssim \sqrt{n/N}$.

Unfortunately, when one has a weaker moment estimate than a ψ_2 one, the situation becomes considerably more difficult. The complexity of the set one has to control remains the same, but the individual concentration deteriorates, because $N^{-1} \sum \langle X_i, x \rangle^2$ does not exhibit a strong enough concentration around its mean to balance the concentration-complexity tradeoff

at the level of $\sqrt{n/N}$. Therefore, with a weaker moment assumption than a ψ_2 one, a combination of individual tail bounds and a “global” assumption, like the small diameter information, is required in both parts of Question 1.2.

One situation in which the process (1.1) has been studied extensively in the last 15 years is a small diameter, ψ_1 moment assumption. The motivation for considering this situation comes from Asymptotic Geometric Analysis and the theory of log-concave measures, which are measures that have a symmetric, log-concave density. They fit the framework at hand nicely, because an isotropic, log concave vector X satisfies that $\|X\|_{\ell_p^n} \leq c_1 n^{1/p}$ with probability at least $1 - 2\exp(-c_2 n^{1/p})$. Indeed, the case $p = 2$ was proved in [24], while for $p > 2$ the result was recently established by Latała in [19]. Moreover, linear functionals exhibit a ψ_1 behavior (see, e.g. [12] for a survey on log-concavity).

Partial results in the isotropic, log-concave case have been obtain by Bourgain [9], yielding an estimate on the covariance operator for $N = c(\varepsilon)n \log^3 n$, which was improved by Rudelson [27] to $N = c(\varepsilon)n \log^2 n$. Subsequent improvements were $N = c(\varepsilon)n \log n$ for unconditional convex bodies in [13] and for general log-concave measures in [24]. Finally, the optimal estimate of $N = c(\varepsilon)n$ was obtained for an unconditional, log-concave measures by Aubrun [4], and for an arbitrary log-concave measure in Adamczak et al. [1, 2], where the following result was proved:

Theorem 1.4 *There exist absolute constants c_1 and c_2 for which the following holds. If μ is an isotropic, log-concave measure, then with probability at least $1 - \exp(-c_1 \sqrt{n})$,*

$$\sup_{t \in S^{n-1}} \left| \frac{1}{N} \sum_{i=1}^N \langle X_i, t \rangle^2 - 1 \right| \leq c_2 \sqrt{\frac{n}{N}}.$$

Naturally, Question 1.2 becomes even harder when one assumes that linear functionals have heavy tails, because sums of independent random variable exhibit very limited concentration – far below the level required for the proof of Theorem 1.4. Recently, Vershynin [35] proved the following remarkable fact:

Theorem 1.5 *For every $q > 4$, $\delta > 0$ and constants κ_1 and κ_2 , there exist constants c_1 and c_2 that depend on q , δ and κ_1, κ_2 for which the following holds.*

If μ satisfies a 2-small diameter, L_q moment assumption with constants κ_1 and κ_2 , then for every $\delta > 0$, with probability at least $1 - \delta$,

$$\|\Sigma_N - \Sigma\|_{2 \rightarrow 2} \leq c_1 (\log \log n)^2 \left(\frac{n}{N}\right)^{1/2-2/q}.$$

In particular, if μ is isotropic then

$$1 - c_2 \left(\frac{n}{N}\right)^{1/2-2/q} (\log \log n)^2 \leq s_{\min}(\Gamma) \leq s_{\max}(\Gamma) \leq 1 + c_2 \left(\frac{n}{N}\right)^{1/2-2/q} (\log \log n)^2.$$

Moreover, very recently Strivastava and Vershynin [29], obtained the following result:

Theorem 1.6 *For every $\eta > 0$, $\varepsilon > 0$ and $\kappa > 0$ there exists constants c_1, c_2 and $c_3 = \frac{\eta}{2\eta+2}$ for which the following holds. Let μ be an isotropic measure, satisfying that for every projection P in \mathbb{R}^n ,*

$$(*) \quad \Pr\{\|PX\|_2^2 > t\} \leq \frac{\kappa}{t^{1+\eta}}, \text{ for } t \geq \kappa \text{ rank}(P).$$

If $(X_i)_{i=1}^N$ are independent random vectors distributed according to μ then for every $N \geq c_1 n$,

$$\mathbb{E}\|\Sigma_N - I_d\| \leq \varepsilon.$$

Moreover, only under a q -moment assumption,

$$1 - c_2 \left(\frac{n}{N}\right)^{c_3} \leq \mathbb{E}s_{\min}(\Gamma)$$

It should be noted that the boundedness assumption in Theorem 1.6 is satisfied by a vector with independent components $X = (\xi_i)_{i=1}^n$, if $\xi \in L_q$ for $q > 4$, and thus both parts may be used in the i.i.d situation. However, for any $\eta > 0$, $c_3 < \frac{1}{2}$ ($1/2$ being the power in the Bai-Yin Theorem).

Our main result gives a version of Theorem 1.5 for an unconditional measure with “heavy tails”.

Theorem A. Let μ be an unconditional measure that satisfies the p -small diameter, L_q moment assumption with constants κ_1 and κ_2 for some $p > 2$.

1. For every $q > 4$ and $\delta < 1/2 - 1/2(p-1)$, there exist constants c_0 , c_1 and c_2 that depend on q , p , κ_1 , κ_2 and δ , such that, for every $n \leq N \leq \exp(c_0 n^\delta)$, with probability at least $1 - \exp(-c_1 n^\delta)$,

$$\sup_{t \in B_2^n} |N^{-1} \sum_{i=1}^N \langle X_i, t \rangle^2 - \mathbb{E} \langle X, t \rangle^2| \leq c_2 \left(\frac{n}{N}\right)^{1/2}.$$

2. For every $2 < q \leq 4$, if $p > (1 - 2/q)^{-1}$ and $\delta < 1/2 - 1/2(p - 1)$, there exist constants c_3 and c_4 that depend on q, p, δ, κ_1 and κ_2 , such that, for every $n \leq N \leq \exp(c_0 n^\delta)$, with probability at least $1 - \exp(-c_3 n^\delta)$,

$$\sup_{t \in B_2^n} |N^{-1} \sum_{i=1}^N \langle X_i, t \rangle^2 - \mathbb{E} \langle X, t \rangle^2| \leq c_4 \left(\frac{n}{N} \right)^{1-2/q} \log(N/n).$$

In both cases, for every $\varepsilon > 0$, with probability at least $1 - 2\exp(-cn^\delta)$, $\|\Sigma_N - \Sigma\|_{2 \rightarrow 2} \leq \varepsilon$ provided that $N \gtrsim_{q,p,\delta,\kappa_1,\kappa_2} n$. Moreover, if μ is isotropic and $q > 4$, then

$$1 - c_2 \left(\frac{n}{N} \right)^{1/2} \leq s_{\min}(\Gamma) \leq s_{\max}(\Gamma) \leq 1 + c_2 \left(\frac{n}{N} \right)^{1/2},$$

and if $2 < q \leq 4$ then

$$1 - c_4 \left(\frac{n}{N} \right)^{1-2/q} \log(N/n) \leq s_{\min}(\Gamma) \leq s_{\max}(\Gamma) \leq 1 + c_4 \left(\frac{n}{N} \right)^{1-2/q} \log(N/n).$$

Our quantitative version of the Bai-Yin Theorem follows from Theorem A, because of the straightforward observation that if $\xi \in L_q$ for $q > 4$ and is symmetric, then $X = (\xi_i)_{i=1}^n$ is unconditional, and there is some $p > 2$ for which $\max_{i \leq N} \|X\|_{\ell_p^n} \lesssim n^{1/p}$ with high enough probability. Thus, conditioning μ to the unconditional body $cn^{1/p}B_p^n$ yield the desired result.

The approach we take in the proof of Theorem A is very different from all the previous results mentioned above, as those rely heavily on the fact that the empirical process (1.1) is indexed by the sphere or by the Euclidean ball, and that the underlying class of functions consists of linear functionals. At the heart of the arguments are either the classical trace method [4], a non-commutative Khintchine inequality [27] or sharp estimates on $\max_{|I|=k} \|\sum_{i \in I} X_i\|_{\ell_2^n}$ [9, 1, 35]. As such, all these proofs are “Euclidean” in nature and can not lead to bounds on the empirical process

$$\sup_{h \in H} \left| \frac{1}{N} \sum_{i=1}^N h^2(X_i) - \mathbb{E} h^2 \right| \quad (1.4)$$

for an arbitrary class of functions H – not even for $H_T = \{\langle t, \cdot \rangle : t \in T\}$ when T is not the sphere or close to the sphere in some sense.

One should note that process (1.4) is an interesting object in its own right. For example, it has a key role in analyzing the uniform central limit

Theorem [10]; and, when indexed by H_T for $T \subset \mathbb{R}^n$, it appear naturally in Asymptotic Geometric Analysis, for example, when proving embedding results or “low- M^* ” estimates for various matrix ensembles (see [22] for a more detailed discussion). Thus, understanding what governs (1.4), and in particular, going beyond the case $H_{B_2^n}$ is rather important.

The proof of Theorem A does just that, since it is based on a bound on (1.4) in terms of a certain notion of “complexity” of the class H . It is not tailored to the case $H_{B_2^n}$, nor does it rely on the fact that the indexing class consists of linear functionals. Rather, the proof is based on a chaining scheme which is much more general than the applications that will be presented here.

The second application we chose to present as an illustration of the potential this empirical processes based method has, is the following.

Let y_1, \dots, y_n be independent, standard exponential random variables (i.e., with density $\sim \exp(-\sqrt{2}|t|)$), and for every $T \subset \mathbb{R}^n$ set

$$E(T) = \mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i y_i, \quad d_2(T) = \sup_{t \in T} \|t\|_{\ell_2^n}.$$

Theorem B. There exists absolute constants c_1 , c_2 and c_3 for which the following holds. If μ is an isotropic, unconditional, log-concave measure on \mathbb{R}^n and $T \subset \mathbb{R}^n$ is centrally symmetric, then for every $u \geq c_1$, with probability at least $1 - 2\exp(-c_2 u^2)$,

$$\sup_{t \in T} \left| \frac{1}{N} \sum_{i=1}^N \langle t, X_i \rangle^2 - \|t\|_{\ell_2^n}^2 \right| \leq c_3 u^3 \left(\frac{E(T)}{\sqrt{N}} + \frac{(E(T))^2}{N} \right). \quad (1.5)$$

To put Theorem B in the right context, recall that a symmetric measure ν on \mathbb{R}^n (κ, L)-weakly dominates a symmetric measure μ if for every $x \in \mathbb{R}^n$, and every $t > 0$, $Pr_\mu(|\langle x, \cdot \rangle| \geq Lt) \leq \kappa Pr_\nu(|\langle x, \cdot \rangle| \geq t)$ [16]. For example, if μ is an isotropic L -subgaussian measure and $G = (g_1, \dots, g_n)$ is a standard gaussian vector in \mathbb{R}^n then

$$Pr_\mu(|\langle x, \cdot \rangle| \geq Lt) \leq 2\exp(-t^2/2\|x\|_{\ell_2^n}^2) = Pr_G(|\langle x, \cdot \rangle| \geq t),$$

and thus μ is weakly dominated by G .

By the Majorizing Measures Theorem (see, e.g., [32] and Section 2), it follows that if μ is L -subgaussian, there is a constant $c = c(L)$ satisfying that for every $T \subset \mathbb{R}^n$ and every integer N ,

$$\mathbb{E} \sup_{t \in T} \left\langle \sum_{i=1}^N X_i, t \right\rangle \leq c \mathbb{E} \sup_{t \in T} \left\langle \sum_{i=1}^N G_i, t \right\rangle \equiv c\sqrt{N}G(T) \quad (1.6)$$

where $(X_i)_{i=1}^N$ are independent copies of X , $(G_i)_{i=1}^N$ are independent copies of G and $G(T) = \mathbb{E} \sup_{t \in T} \langle G, t \rangle$.

Moreover, the results of [21, 22] show that if T is centrally symmetric and μ is isotropic and L -subgaussian, then

$$\mathbb{E} \sup_{t \in T} \left| \frac{1}{N} \sum_{i=1}^N \langle t, X_i \rangle^2 - \|t\|_{\ell_2^n}^2 \right| \lesssim_L \frac{G(T)}{\sqrt{N}} + \frac{(G(T))^2}{N}. \quad (1.7)$$

Hence, the fact that an L -subgaussian measure is weakly dominated by a gaussian measure (with the same covariance structure) is exhibited by a strong domination in (1.6) and in (1.7), that holds for every $T \subset \mathbb{R}^n$.

Just like subgaussian vectors, isotropic, unconditional log-concave vectors have a natural weakly dominating measure. By the Bobkov-Nazarov Theorem [7] they are (κ, L) -weakly dominated by the vector $Y = (y_1, \dots, y_n)$, and κ and L are absolute constants. In [18], Latała showed that as in (1.6), for every $T \subset \mathbb{R}^n$, $\mathbb{E} \sup_{t \in T} \langle \sum_{i=1}^N X_i, t \rangle \lesssim \mathbb{E} \sup_{t \in T} \langle \sum_{i=1}^N Y_i, t \rangle$. Theorem B shows that the quadratic strong domination, analogous to (1.7), is also true in this case.

Theorem B has many standard applications, leading to embedding results of a similar nature to the Johnson-Lindenstrauss Lemma and to “low M^* ” estimates that hold for unconditional, log-concave ensembles. Deriving these and other outcomes from Theorem B is standard and will not be presented here. One should also note that a log-concave Chevet type inequality, i.e., upper estimates on the operator norm $\|\Gamma\|_{X \rightarrow Y}$ for finite dimensional normed spaces X and Y has recently been established in [3].

In the next section we will present several preliminary facts and definitions that will be used throughout this article. Then, in Section 3 we will show that if $V \subset \mathbb{R}^N$ can be decomposed in a certain way, the Bernoulli process indexed by $\{(v_i^2)_{i=1}^N : v \in V\}$ is well behaved. Section 4 is devoted to the observation that if H is a class of functions, then under mild assumptions and with high probability, the random coordinate projection $P_\sigma H = \{(h(X_i))_{i=1}^N : h \in H\}$ can be decomposed in the sense of Section 3. It turns out that the decomposition depends on the complexity of H and on the decay of tails of functions in H . Finally, in Section 5 we will present examples in which the complexity of H can be estimated, leading to the proofs of Theorem A (and consequently, the quantitative Bai-Yin Theorem) and of Theorem B.

2 Preliminaries

Throughout, all absolute constants are positive numbers, denoted by c, c_0, c_1, \dots and their value may change from line to line. $\kappa_0, \kappa_1, \dots$ denote constants whose value will remain unchanged. By $A \sim B$ we mean that there are absolute constants c and C such that $cB \leq A \leq CB$, and by $A \lesssim B$ that $A \leq CB$. $A \sim_\gamma B$ (resp. $A \lesssim_\gamma B$) denotes that the constants depend only on γ .

For $1 \leq p \leq \infty$, ℓ_p^n is \mathbb{R}^n endowed with the ℓ_p norm, which we denote by $\|\cdot\|_{\ell_p^n}$, and B_p^n is its unit ball. With a minor abuse of notation we write $|\cdot|$ both for the cardinality of a set and for the absolute value. Finally, if (a_n) is a sequence, let (a_n^*) be a non-increasing rearrangement of $(|a_n|)$.

Next, let us turn to the complexity parameters that motivated our method of analysis – Talagrand’s γ -functionals.

Definition 2.1 [32] *For a metric space (T, d) , an admissible sequence of T is a collection of subsets of T , $\{T_s : s \geq 0\}$, such that for every $s \geq 1$, $|T_s| \leq 2^{2^s}$ and $|T_0| = 1$. For $\beta \geq 1$, define the γ_β functional by*

$$\gamma_\beta(T, d) = \inf \sup_{t \in T} \sum_{s=0}^{\infty} 2^{s/\beta} d(t, T_s),$$

where the infimum is taken with respect to all admissible sequences of T . For an admissible sequence $(T_s)_{s \geq 0}$ we denote by $\pi_{s,t}$ a nearest point to t in T_s with respect to the metric d .

One should note that our chaining approach is based on a slightly less restrictive definition, giving one more freedom; for example, the cardinality of the sets will not necessarily be 2^{2^s} , the metric may change with s , etc. (see Section 3).

When considered for a set $T \subset L_2$, γ_2 has close connections with properties of the canonical gaussian process indexed by T , and we refer the reader to [10, 32] for detailed expositions on these connections. One can show that under mild measurability assumptions, if $\{G_t : t \in T\}$ is a centered gaussian process indexed by a set T , then

$$c_1 \gamma_2(T, d) \leq \mathbb{E} \sup_{t \in T} G_t \leq c_2 \gamma_2(T, d),$$

where c_1 and c_2 are absolute constants and for every $s, t \in T$, $d^2(s, t) = \mathbb{E}|G_s - G_t|^2$. The upper bound is due to Fernique [11] and the lower bound is Talagrand’s Majorizing Measures Theorem [30]. Note that if $T \subset \mathbb{R}^n$, $(g_i)_{i=1}^n$

are standard, independent gaussians and $G_t = \sum_{i=1}^n g_i t_i$ then $d(s, t) = \|s - t\|_{\ell_2^n}$, and therefore

$$c_1 \gamma_2(T, \|\cdot\|_{\ell_2^n}) \leq \mathbb{E} \sup_{t \in T} \sum_{i=1}^n g_i t_i \leq c_2 \gamma_2(T, \|\cdot\|_{\ell_2^n}). \quad (2.1)$$

A part of our discussion (Theorem B) will be devoted to isotropic, log-concave measures on \mathbb{R}^n .

Definition 2.2 *A symmetric probability measure μ on \mathbb{R}^n is isotropic if for every $y \in \mathbb{R}^n$, $\int |\langle x, y \rangle|^2 d\mu(x) = \|y\|_{\ell_2^n}^2$.*

The measure μ is log-concave if for every $0 < \lambda < 1$ and every nonempty Borel measurable sets $A, B \subset \mathbb{R}^n$, $\mu(\lambda A + (1 - \lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}$.

A typical example of a log-concave measure on \mathbb{R}^n is the volume measure of a convex body in \mathbb{R}^n , a fact that follows from the Brunn-Minkowski inequality (see, e.g. [26]). Moreover, Borell's inequality [8, 23] implies that there is an absolute constant c such that if μ is an isotropic, log-concave measure on \mathbb{R}^n , then for every $x \in \mathbb{R}^n$, $\|\langle x, \cdot \rangle\|_{\psi_1} \leq c \|\langle x, \cdot \rangle\|_{L_2} = c \|x\|_{\ell_2^n}$.

As mentioned in the introduction, if X is distributed according to an isotropic, log-concave measure on \mathbb{R}^n then $\|X\|_{\ell_p^n}$ decays quickly at scales that are larger than $n^{1/p}$. Thus, by conditioning, the main result in [24] shows that a 2-small diameter assumption can be made without loss of generality as long as $N \leq \exp(c\sqrt{n})$, and Latała [19] proved the analogous result for $p > 2$, as long as $N \leq \exp(cn^{1/p})$.

3 Decomposition of sets

We begin with a description of the modified chaining procedure. Let $(\eta_s)_{s \geq 0}$ be an increasing sequence which satisfies that for every $s \geq 0$, $2^{\eta_s} \cdot 2^{\eta_{s+1}} \leq 10 \cdot 2^{\eta_{s+2}}$ and for $s \geq 1$, $1.1 \leq \eta_{s+1}/\eta_s \leq 10$ (where 1.1 can be replaced by $1 + \varepsilon$ and 10 can be any suitably large constant). For example, $\eta_0 = 0$ and $\eta_s = 2^s$ for $s \geq 1$ is the usual choice of a sequence that has been used in the definition of Talagrand's γ functionals. An admissible sequence of $V \subset \mathbb{R}^N$ relative to $(\eta_s)_{s \geq 0}$ is a collection of subsets $V_s \subset V$ for which $|V_s| \leq 2^{\eta_s}$. For every s let $\pi_s : V \rightarrow V_s$, which usually will be a nearest point map relative to some distance. We will denote $\pi_s v - \pi_{s-1} v$ by $\Delta_s v$, and sometimes write $\Delta_0 v$ for $\pi_0 v$. Finally, $\Delta_s V$ is the set $\{\Delta_s v : v \in V\}$.

Let ϕ be an increasing function which will be chosen according to additional information one will have on the given class. Examples that one

should have in mind are $\phi_\beta(x) \sim_\beta \sqrt{x} \log^{1/\beta}(eN/x)$, resulting from a bound on the ψ_β diameter of H , or $\phi_{q,\varepsilon} \sim_{q,\varepsilon} N^{(1+\varepsilon)/q} x^{1/2-(1+\varepsilon)/q}$ for $q > 2$ and ε in the right range, arising from an L_q moment assumption.

Assume that $V \subset \mathbb{R}^N$ is endowed with a family of functionals θ_s and a semi-norm $\| \cdot \|$ (which, in our applications, will either arise from the L_q norm or from the ψ_β norm), and set $d = \sup_{v \in V} \|v\|$.

Definition 3.1 $V \subset \mathbb{R}^N$ admits a decomposition with constants α and γ if it has an admissible sequence $(V_s)_{s \geq 0}$ relative to $(\eta_s)_{s \geq 0}$ for which the following holds.

1. $\sup_{v \in V} (\theta_0(\pi_0 v) + \sum_{s > 0} \theta_s(\Delta_s v)) \leq \gamma$.

2. For every $v \in V$ and every $I \subset \{1, \dots, N\}$,

$$\left(\sum_{i \in I} v_i^2 \right)^{1/2} \leq \alpha (\gamma + d\phi(|I|)).$$

3. If $\eta_s \leq N$ then for every $v \in V$ and every $I \subset \{1, \dots, N\}$

$$\left(\sum_{i \in I} (\Delta_s v)_i^2 \right)^{1/2} \leq \alpha (\theta_s(\Delta_s v) + \|\Delta_s v\| \phi(|I|)),$$

and if $\eta_s \geq N$ then for every $v \in V$ and every $I \subset \{1, \dots, N\}$,

$$\left(\sum_{i \in I} (\Delta_s v)_i^2 \right)^{1/2} \leq \alpha \theta_s(\Delta_s v).$$

Although this definition seems artificial at first glance, we will show that it captures the geometry of a typical coordinate projection $P_\sigma H = \{(h(X_i))_{i=1}^N : h \in H\}$.

The main observation of this section is that one can use this type of decomposition to bound the supremum of the Bernoulli process indexed by $V^2 = \{(v_i^2)_{i=1}^N : v \in V\}$. Hence, if $V = P_\sigma H$, then a standard symmetrization argument leads to the desired bound on $\sup_{h \in H} |N^{-1} \sum_{i=1}^N h^2(X_i) - \mathbb{E} h^2|$ (see section 5.3).

To formulate the estimate on the Bernoulli process, set

$$\Phi = \left(\sum_{i=1}^N \frac{\phi^4(i)}{i^2} \right)^{1/2}, \quad \Phi_s = \left(\sum_{i=1}^{N-\eta_s} \frac{\phi^2(\eta_s + i)}{\eta_s + i} \cdot \frac{\phi^2(i)}{i} \right)^{1/2}$$

for $\eta_s \leq N$, put

$$A_1 = \sup_{v \in V} \sum_{\{s > 0: \eta_s \leq N\}} \phi(\eta_s) \|\Delta_s v\|, \quad A_2 = \sup_{v \in V} \sum_{\{s > 0: \eta_s \leq N\}} \phi^2(\eta_s) \|\Delta_s v\|,$$

and let

$$A_\Phi = \sup_{v \in V} \sum_{\{s: \eta_s \leq N\}} \Phi_s \eta_s^{1/2} \|\Delta_s v\|.$$

For $2 < q \leq 4$ and $0 \leq \varepsilon < (q/2) - 1$, let

$$B_{q,\varepsilon} = \sup_{v \in V} \sum_{\{s: \eta_s \leq N\}} \eta_s^{1-2(1+\varepsilon)/q} \|\Delta_s v\|.$$

As will become clearer, the most important of the $B_{q,\varepsilon}$ parameters is

$$B_4 \equiv B_{4,0} = \sup_{v \in V} \sum_{\{s: \eta_s \leq N\}} \eta_s^{1/2} \|\Delta_s v\|,$$

which, under the standard choice of $\eta_0 = 0$ and $\eta_s = 2^s$ for $s \geq 1$, corresponds to $\gamma_2(V, \|\cdot\|)$.

Theorem 3.2 *There exist absolute constants c_0 , c_1 and c_2 for which the following holds. If $V \subset \mathbb{R}^N$ has a decomposition as in Definition 3.1, then for every $r \geq c_0$, with probability at least $1 - 2 \exp(-c_1 r^2 \eta_0)$,*

$$\sup_{v \in V} \left| \sum_{i=1}^N \varepsilon_i v_i^2 \right| \leq c_2 r \alpha^2 (\gamma(\gamma + d\phi(N) + A_1) + d(A_2 + A_\Phi))$$

Before presenting the proof, let us consider the two main examples which will interest us, namely, the families $\phi_\beta = \sqrt{x} \log^{1/\beta}(eN/x)$ for any $\beta > 0$ and $\phi_{q,\varepsilon} = \sqrt{x}(N/x)^{(1+\varepsilon)/q}$ for any $q > 2$ (and for ε selected appropriately).

In both cases $\phi(N) \sim \sqrt{N}$ and for any $\beta > 0$, $\Phi \sim_\beta \sqrt{N}$. If $q > 4$ and $0 \leq \varepsilon \leq q/4 - 1$, $\Phi \leq (1 - 4(1 + \varepsilon)/q)^{-1/2} \sqrt{N}$, and since $\Phi_s \leq \Phi$, then for $\beta > 0$ or $q > 4$,

$$A_\Phi \leq \Phi \sup_{v \in V} \sum_{\{s: \eta_s \leq N\}} \eta_s^{1/2} \|\Delta_s v\| \lesssim \sqrt{N} B_4,$$

with the constant depending either on β or on q and ε as above.

On the other hand, if $2 < q \leq 4$ and $0 < \varepsilon < q/2 - 1$ then

$$\begin{aligned}\Phi_s &= \left(\sum_{i=1}^{N-\eta_s} \frac{\phi^2(\eta_s + i)}{\eta_s + i} \cdot \frac{\phi^2(i)}{i} \right)^{1/2} = \left(\sum_{i=1}^{N-\eta_s} \left(\frac{N}{\eta_s + i} \right)^{2(1+\varepsilon)/q} \cdot \left(\frac{N}{i} \right)^{2(1+\varepsilon)/q} \right)^{1/2} \\ &\lesssim \frac{N^{2(1+\varepsilon)/q}}{1 - 2(1+\varepsilon)/q} \cdot \eta_s^{1/2 - 2(1+\varepsilon)/q}.\end{aligned}$$

Therefore, in that range

$$A_\Phi \lesssim \frac{N^{2(1+\varepsilon)/q}}{1 - 2(1+\varepsilon)/q} \cdot \sup_{v \in V} \sum_{\{s: \eta_s \leq N\}} \eta_s^{1-2(1+\varepsilon)/q} \|\Delta_s v\| = \frac{N^{2(1+\varepsilon)/q}}{1 - 2(1+\varepsilon)/q} B_{q,\varepsilon}.$$

Next, since $(\eta_s)_{s \geq 0}$ increases exponentially, then for $q > 2$

$$\sum_{\{s: \eta_s \leq N\}} \phi(\eta_s) \|\Delta_s v\| \leq 2d \sum_{\{s: \eta_s \leq N\}} \phi(\eta_s) \lesssim d\sqrt{N}, \quad (3.1)$$

and the constant in (3.1) depends on β or on q and ε respectively. In particular, if $2 < q \leq 4$ and $0 < \varepsilon < q/2 - 1$, then

$$\sum_{\{s: \eta_s \leq N\}} \phi(\eta_s) \lesssim \frac{\sqrt{N}}{1 - 2(1+\varepsilon)/q}$$

Finally, one has to control $\sum_{\{s: \eta_s \leq N\}} \phi^2(\eta_s) \|\Delta_s v\|$. Note that if $\beta > 0$ or $q > 4$, then

$$\begin{aligned}\sum_{\{s: \eta_s \leq N\}} \phi^2(\eta_s) \|\Delta_s v\| &\leq \left(\max_{\{s: \eta_s \leq N\}} \frac{\phi^2(\eta_s)}{\eta_s^{1/2}} \right) \cdot \sum_{\{s: \eta_s \leq N\}} \eta_s^{1/2} \|\Delta_s v\| \\ &\lesssim \sqrt{N} \sum_{\{s: \eta_s \leq N\}} \eta_s^{1/2} \|\Delta_s v\| \sim \sqrt{N} B_4,\end{aligned}$$

and if $2 < q \leq 4$ and $0 < \varepsilon < q/2 - 1$ then

$$\sum_{\{s: \eta_s \leq N\}} \phi^2(\eta_s) \|\Delta_s v\| \leq N^{2(1+\varepsilon)/q} \sum_{\{s: \eta_s \leq N\}} \eta_s^{1-2(1+\varepsilon)/q} \|\Delta_s v\| = N^{2(1+\varepsilon)/q} B_{q,\varepsilon}.$$

We thus arrive to a more compact formulation of Theorem 3.2 in the cases we will be interested in.

Corollary 3.3 For any $\beta > 0$ or $q > 4$, with probability at least $1 - 2\exp(-c_1 r^2 \eta_0)$,

$$\sup_{v \in V} \left| \sum_{i=1}^N \varepsilon_i v_i^2 \right| \lesssim r \alpha^2 \left(\gamma^2 + d\sqrt{N}(\gamma + B_4) \right),$$

with a constant that depends on β or on q and ε respectively.

Also, if $2 < q \leq 4$ and $0 < \varepsilon < q/2 - 1$, then with probability at least $1 - 2\exp(-c_1 r^2 \eta_0)$,

$$\sup_{v \in V} \left| \sum_{i=1}^N \varepsilon_i v_i^2 \right| \lesssim \frac{\alpha^2 r}{1 - 2(1 + \varepsilon)/q} \left(\gamma^2 + d\sqrt{N}\gamma + dN^{2(1+\varepsilon)/q} B_{q,\varepsilon} \right).$$

Proof of Theorem 3.2. For every $\Delta_s v$ let i be the largest integer in $\{1, \dots, N\}$ for which $\theta_s(\Delta_s v) \geq \|\Delta_s v\| \phi(i)$. Throughout the proof we will assume that such an integer exists, and if it does not, the necessary modifications to the proof are obvious. Let $i_{s,v} = \max\{i, \eta_s\}$ and put $I_{s,v}$ to be the set of the largest $i_{s,v}$ coordinates of $|\Delta_s v|$. Let $\Delta_s^+ v = P_{I_{s,v}} \Delta_s v$ and $\Delta_s^- v = P_{I_{s,v}^c} \Delta_s v$ be the projections of $\Delta_s v$ onto the set of coordinates $I_{s,v}$ and $I_{s,v}^c$ respectively. Also, let j be the largest integer in $\{1, \dots, N\}$ for which $\gamma \geq d\phi(j)$. Thus, for every $v \in V$, $\left(\sum_{i=1}^j (v_i^2)^* \right)^{1/2} \leq 2\alpha\gamma$ and for every $\ell \geq j$, $v_\ell^* \leq 2\alpha d\phi(\ell)/\sqrt{\ell}$. If J is the set of the largest j coordinates of $v \in V$, let $v^+ = P_J v$ and $v^- = P_{J^c} v$.

Let $w \cdot v = \sum_{i=1}^N w_i v_i e_i$, and since

$$v^2 - (\pi_0 v)^2 = \sum_{s>0} (\pi_s v)^2 - (\pi_{s-1} v)^2 = \sum_{s>0} (\Delta_s v) \cdot (\pi_s v + \pi_{s-1} v),$$

one has to control increments of the form $\sum_{i=1}^N \varepsilon_i (\Delta_s v)_i (\pi_s v + \pi_{s-1} v)_i$.

Observe that if $\eta_s \geq N$ then with probability 1,

$$\begin{aligned} \sum_{i=1}^N \varepsilon_i ((\Delta_s v) \cdot (\pi_s v + \pi_{s-1} v))_i &\leq \|(\Delta_s v) \cdot (\pi_s v + \pi_{s-1} v)\|_{\ell_1^N} \\ &\leq 2\|\Delta_s v\|_{\ell_2^N} \sup_{v \in V} \|v\|_{\ell_2^N} \leq 2\alpha^2 \theta_s(\Delta_s v) (\gamma + d\phi(N)). \end{aligned}$$

Next, if $\eta_s \leq N$ we will decompose the vectors one has to control according

to the size of their coordinates, because, with probability $1 - 2\exp(-r^2/2)$,

$$\begin{aligned} \left| \sum_{i=1}^N \varepsilon_i (\Delta_s v)_i (\pi_s v + \pi_{s-1} v)_i \right| &\leq \|(\Delta_s^+ v) \cdot (\pi_s v + \pi_{s-1} v)\|_{\ell_1^N} \\ &+ r \|(\Delta_s^- v) \cdot ((\pi_s v)^+ + (\pi_{s-1} v)^+)\|_{\ell_2^N} + r \|(\Delta_s^- v) \cdot ((\pi_s v)^- + (\pi_{s-1} v)^-)\|_{\ell_2^N}. \end{aligned} \quad (3.2)$$

Consider the following two cases. If $i_{s,v} = \eta_s$ then

$$\begin{aligned} \|(\Delta_s^+ v) \cdot (\pi_s v + \pi_{s-1} v)\|_{\ell_1^N} &\leq \|\Delta_s^+ v\|_{\ell_2^N} \|P_{I_{s,v}}(\pi_s v + \pi_{s-1} v)\|_{\ell_2^N} \\ &\lesssim_{\alpha^2} \|\Delta_s v\| \phi(\eta_s) (\gamma + d\phi(\eta_s)). \end{aligned}$$

Moreover,

$$\|\Delta_s^- v\|_{\ell_\infty^N} \leq \frac{\|\Delta_s^+ v\|_{\ell_2^N}}{|I_{s,v}|^{1/2}} \leq 2\alpha \|\Delta_s v\| \frac{\phi(\eta_s)}{\eta_s^{1/2}},$$

and thus, for every $v \in V$,

$$\eta_s^{1/2} \|(\Delta_s^- v) \cdot w^+\|_{\ell_2^N} \leq \eta_s^{1/2} \|\Delta_s^- v\|_{\ell_\infty^N} \|w^+\|_{\ell_2^N} \lesssim_{\alpha^2} \gamma \phi(\eta_s) \|\Delta_s v\|.$$

To estimate $\eta_s^{1/2} \|(\Delta_s^- v) \cdot w^-\|_{\ell_2^N}$, observe that since $(\Delta_s^- v)_i^* \lesssim_{\alpha} \|\Delta_s v\| \phi(\eta_s + i)/\sqrt{\eta_s + i}$, $w_i^* \lesssim_{\alpha} d\phi(i)/\sqrt{i}$ and $\sum |a_i b_i| \leq \sum a_i^* b_i^*$, then

$$\begin{aligned} \eta_s^{1/2} \|(\Delta_s^- v) \cdot w^-\|_{\ell_2^N} &\lesssim_{\alpha^2} \eta_s^{1/2} \|\Delta_s v\| d \left(\sum_{i=1}^{N-\eta_s} \frac{\phi^2(i)}{i} \cdot \frac{\phi^2(\eta_s + i)}{\eta_s + i} \right)^{1/2} \\ &\lesssim_{\alpha^2} d \eta_s^{1/2} \Phi_s \|\Delta_s v\|. \end{aligned}$$

Therefore, summing the three terms over $\{s > 0 : \eta_s \leq N\}$,

$$\begin{aligned} &\sum_{\{s>0:\eta_s \leq N\}} \|(\Delta_s^+ v) \cdot (\pi_s v + \pi_{s-1} v)\|_{\ell_1^N} \\ &\lesssim_{\alpha^2} \gamma \sum_{\{s>0:\eta_s \leq N\}} \phi(\eta_s) \|\Delta_s v\| + d \sum_{\{s>0:\eta_s \leq N\}} \phi^2(\eta_s) \|\Delta_s v\|, \\ &\sum_{\{s>0:\eta_s \leq N\}} \eta_s^{1/2} \|(\Delta_s^- v) \cdot ((\pi_s v)^+ + (\pi_{s-1} v)^+)\|_{\ell_2^N} \lesssim_{\alpha^2} \gamma \sum_{\{s>0:\eta_s \leq N\}} \phi(\eta_s) \|\Delta_s v\|, \end{aligned}$$

and

$$\sum_{\{s>0:\eta_s \leq N\}} \eta_s^{1/2} \|(\Delta_s^- v) \cdot ((\pi_s v)^- + (\pi_{s-1} v)^-)\|_{\ell_2^N} \lesssim_{\alpha^2} d \sum_{\{s>0:\eta_s \leq N\}} \eta_s^{1/2} \Phi_s \|\Delta_s v\|.$$

Next, if $i_{s,v} \neq \eta_s$ then $\|\Delta_s^+ v\|_{\ell_2^N} \leq 2\alpha\theta_s(\Delta_s v)$, and thus

$$\|(\Delta_s^+ v) \cdot (\pi_s v + \pi_{s-1} v)\|_{\ell_1^N} \leq 2\alpha^2 \theta_s(\Delta_s v)(\gamma + d\phi(N)).$$

Since $|I_{s,v}| \geq \eta_s$,

$$\|\Delta_s^- v\|_{\ell_\infty^N} \leq 2\alpha\theta_s(\Delta_s v)/|I_{s,v}|^{1/2} \leq 2\alpha \frac{\theta_s(\Delta_s v)}{\eta_s^{1/2}},$$

then splitting each $w \in V$ to $w^+ + w^-$ as above,

$$\eta_s^{1/2} \|(\Delta_s^- v) \cdot w^+\|_{\ell_2^N} \leq \eta_s^{1/2} \|\Delta_s^- v\|_{\ell_\infty^N} \|w^+\|_{\ell_2^N} \lesssim_{\alpha^2} \gamma \theta_s(\Delta_s v),$$

and

$$\eta_s^{1/2} \|(\Delta_s^- v) \cdot w^-\|_{\ell_2^N} \lesssim_{\alpha^2} d \eta_s^{1/2} \Phi_s \|\Delta_s v\|.$$

Therefore,

$$\begin{aligned} (*) &= \sum_{\{s>0:\eta_s \leq N\}} \|(\Delta_s^+ v) \cdot (\pi_s v + \pi_{s-1} v)\|_{\ell_1^N} + \eta_s^{1/2} \|(\Delta_s^- v) \cdot ((\pi_s v)^+ + (\pi_{s-1} v)^+)\|_{\ell_2^N} \\ &\quad + \eta_s^{1/2} \|(\Delta_s^- v) \cdot ((\pi_s v)^- + (\pi_{s-1} v)^-)\|_{\ell_2^N} \leq (3) + (4) + (5), \end{aligned}$$

where

$$(3) \lesssim_{\alpha^2} (\gamma + d\phi(N)) \sum_{\{s>0:\eta_s \leq N\}} \theta_s(\Delta_s v), \quad (4) \lesssim_{\alpha^2} \gamma \sum_{\{s>0:\eta_s \leq N\}} \theta_s(\Delta_s v),$$

and

$$(5) \lesssim_{\alpha^2} d \sum_{\{s>0:\eta_s \leq N\}} \eta_s^{1/2} \Phi_s \|\Delta_s v\|.$$

Recall that $|\Delta_s V|, |V_s| \leq 10 \cdot 2^{\eta_{s+1}}$ and that $\eta_{s+1} \leq 10\eta_s$. Given $r \geq c_0$, then applying (3.2) for $t_s = 10r\eta_s^{1/2}$ and summing over $\{s : \eta_s \leq N\}$, it follows that $\sup_{v \in V} \left| \sum_{i=1}^N \varepsilon_i (v^2 - (\pi_0 v)^2)_i \right|$ is bounded by the desired quantity with probability at least $1 - 2\exp(-c_1 r^2 \eta_0)$.

Finally, for $v \in V_0$, let i be the largest integer in $\{1, \dots, N\}$ for which $\theta_0(v) \geq \|v\|\phi(i)$, and set I to be the set of the i -largest coordinates of v . Thus, $\sum_{i \in I} v_i^2 \leq 2\alpha^2 \theta_0(v) \leq 2\alpha^2 \gamma^2$, and for $\ell \geq i$, $v_\ell^* \leq \alpha \|v\| \phi(\ell)/\sqrt{\ell}$. Since $|V_0| \leq 2^{\eta_0}$, then with probability at least $1 - 2\exp(-c_2 r^2 \eta_0)$

$$\left| \sum_{i=1}^N \varepsilon_i v_i^2 \right| \lesssim_{\alpha^2} \gamma^2 + rd\Phi_0 \eta_0^{1/2} \|v\|,$$

completing the proof. ■

4 Coordinate projections of Function classes

The aim of this section is to show that under very mild assumptions, empirical processes have well behaved coordinate projections in the sense of Definition 3.1. A first result in this direction was established in [22], in which the main observation, formulated in the language of Section 3, was that if $\eta_0 = 0$ and $\eta_s = 2^s$ for $s \geq 1$, then for the choice of $\theta_s((h(X_i))_{i=1}^N) = 2^{s/2} \|h\|_{\psi_2}$, $\alpha = \sqrt{u}$, $\|(h(X_i))_{i=1}^N\| = \|h\|_{\psi_1}$ and $\phi(x) \sim \sqrt{x} \log(eN/x)$, the set $V = \{(h(X_i))_{i=1}^N : h \in H\}$ has a good decomposition with high probability. Hence, the Bernoulli process indexed by V^2 satisfies the following:

Theorem 4.1 *There exist absolute constants c_1 , c_2 and c_3 for which the following holds. If H is a class of functions, then for every $r, u \geq c_1$, with μ^N -probability at least $1 - 2\exp(-c_2u)$, $V = P_\sigma H$ satisfies that*

$$\sup_{v \in V} \left| \sum_{i=1}^N \varepsilon_i v_i^2 \right| \lesssim ru^2 \left(\gamma_2(H, \psi_2) + \sqrt{N} \sup_{h \in H} \|h\|_{\psi_1} \right) \cdot \gamma_2(H, \psi_2)$$

with probability at least $1 - 2\exp(-c_3r^2)$ with respect to the Bernoulli random variables.

Theorem 4.1 is rather restricted because the ψ_2 -based complexity parameter seems too strong in many situations, as does the assumption that H is a bounded subset of L_{ψ_1} . Here, we will try to impose as few assumptions as possible on H .

Let H be a class of functions on (Ω, μ) . For every $u > 0$ we will define three events in the product space Ω^N , which will be denoted by $\Omega_{1,u}$, $\Omega_{2,u}$ and $\Omega_{3,u}$. On the event $\Omega_{1,u} \cap \Omega_{2,u} \cap \Omega_{3,u}$, the random set $P_\sigma H = \{(h(X_i))_{i=1}^N : h \in H\}$ will be well behaved for the right choice of functionals θ_s and ϕ . We will then study cases in which the event $\Omega_{1,u} \cap \Omega_{2,u} \cap \Omega_{3,u}$ has high probability.

Definition 4.2 *For $(\eta_s)_{s \geq 0}$ as above, set $s_0 \geq 0$ to be the first integer for which $\eta_s \geq \log(eN)$.*

For every $s \in \{s : \log(eN) \leq \eta_s \leq N\}$, let ℓ_s be the largest integer in $\{1, \dots, N\}$ for which $\eta_s \geq \ell \log(eN/\ell)$, and if $\eta_s \leq \log(eN)$, set $\ell_s = 1$.

The motivation for this definition is the following. If E_k is the collection of subsets of $\{1, \dots, N\}$ of cardinality k , s_0 is the level above which one may find k for which the cardinalities $|E_k|$ and $|H_s|$ are comparable. Indeed, when $s < s_0$, $|E_1|$ can be significantly larger than $|H_s| = 2^{\eta_s}$, but when

$s \geq s_0$, $\log |H_s|$ and $\log |E_{\ell_s}|$ are of the same order, and thus one may simultaneously control every function in H_s and every subset in E_{ℓ_s} at no extra price. The main idea of the proofs in this section is to try and balance these two quantities as much as possible.

Observe that since $(\eta_s)_{s=0}^\infty$ grows exponentially, so does $(\ell_s)_{s \geq s_0}$.

Definition 4.3 *For an admissible sequence $(H_s)_{s \geq 0}$ and a sequence of functionals $(\theta_{u,s})_{s \geq s_0}$, let $\Omega_{1,u}$ be the event for which, for every $h \in H$, the following holds:*

1. *for every $\log(eN) \leq \eta_s \leq N$, $\left(\sum_{i=1}^{\ell_{s+1}} ((\Delta_s h)^2(X_i))^*\right)^{1/2} \leq \theta_{u,s}(\Delta_s h)$,
(and if the $\ell_{s+1} > N$ then the sum terminates at N).*
2. *for every $\eta_s > N$, $\left(\sum_{i=1}^N ((\Delta_s h)^2(X_i))^*\right)^{1/2} \leq \theta_{u,s}(\Delta_s h)$.*
3. $\left(\sum_{i=1}^{\ell_{s_0+1}} ((\pi_{s_0} h)^2(X_i))^*\right)^{1/2} \leq \theta_{u,s_0}(\pi_{s_0} h)$.

The set $\Omega_{1,u}$ is the subset of Ω^N in which the functionals $\theta_{u,s}$ yield a good bound on the ℓ_2 norm of the “relatively large” coordinates of each increment when $s \geq s_0$. In contrast, on the set $\Omega_{2,u}$ the smaller coordinates will be controlled for $s \geq s_0$. One of the key points of the proof is finding an estimate on the ℓ_2^n norm on these coordinates, but doing so without any real concentration phenomenon for sums of i.i.d. random variables coming to one’s aid.

Formally, to define the set $\Omega_{2,u}$, first fix a random variable Y , an integer N and $\varepsilon > 0$. For every $j \leq N$ let $\delta_j = (j/eN)^{(1+\varepsilon)}$, set

$$y_j = \inf\{y : \Pr(|Y| \geq y_j) \leq \delta_j\},$$

and without loss of generality, we will assume that the infimum is attained.

For every $1 \leq k \leq N$, let

$$f_u(Y, k) = \kappa_3 \sqrt{u} \left(\sum_{\{j: 2^j \leq \lceil k/u \rceil\}} 2^j y_{2^j}^2 \right)^{1/2},$$

where κ_3 is a suitable chosen absolute constant.

The motivation for this definition is the following observation, showing that with high probability, the “tail” of a sum of i.i.d random variables can be controlled using f .

Lemma 4.4 *There exist absolute constants c_1 and c_2 for which the following holds. For every integer ℓ and $u \geq c_1/\varepsilon$, with probability at least $1 - 2 \exp(-c_2 u \varepsilon \ell \log(eN/\ell))$, for every integer $k > u\ell$,*

$$\sum_{i=u\ell+1}^k (Y_i^2)^* \leq f_u(Y, k)$$

Proof. Since $Pr(|Y| \geq y_j) \leq \delta_j = (j/eN)^{1+\varepsilon}$ then for $u \geq 1$,

$$\begin{aligned} Pr(Y_{uj}^* \geq y_j) &\leq \binom{N}{uj} \delta_j^{uj} \leq \exp(uj \log(eN/uj) - (1+\varepsilon)uj \log(eN/j)) \\ &\leq \exp(-\varepsilon uj \log(eN/j)). \end{aligned}$$

Thus, summing over $\{j = \lceil \ell + 2^i/u \rceil : 2^i \leq k - u\ell\}$, it follows that with probability at least $1 - \exp(-c_1 \varepsilon u \ell \log(eN/\ell))$, if $2^i \leq k - u\ell$ then $Y_{u\ell+2^i}^* \leq y_{\lceil \ell+2^i/u \rceil}$. Therefore,

$$\begin{aligned} \sum_{j=u\ell+1}^k (Y_j^2)^* &\leq \sum_{\{i: 2^i \leq k-u\ell\}} 2^i (Y_{u\ell+2^{i-1}}^2)^* \leq \sum_{\{i: 2^i \leq k-u\ell\}} 2^i y_{\lceil \ell+2^{i-1}/u \rceil}^2 \\ &\leq c_2 u \sum_{\{j: 2^j \leq k/u\}} 2^j y_{2^j}^2 = f^2(Y, k), \end{aligned}$$

where the last inequality is evident by a change of variables. \blacksquare

We will also need the following “global” counterpart of the functional f .

Definition 4.5 *Given a class of functions H , an integer N and $\varepsilon > 0$, set*

$$z_j = \inf\{z : \sup_{h \in H} Pr(|h| \geq z_j) \leq (j/eN)^{1+\varepsilon}\}.$$

For every $k \leq N$ and $u \geq 1$, let

$$F_u(k) = \kappa_3 \sqrt{u} \left(\sum_{\{j: 2^j \leq k/u\}} 2^j z_{2^j}^2 \right)^{1/2}$$

Clearly, for every $h \in H$ and every k , $f_u(h, k) \leq F_u(k)$.

Definition 4.6 *Let $\Omega_{2,u}$ be the event on which, for every $h \in H$, every $s \geq s_0$ and every $j > u\ell_s$*

1. $\left(\sum_{i=u\ell_s+1}^j ((\Delta_s h)^2(X_i))^*\right)^{1/2} \leq f_u(\Delta_s h, j),$
2. $\left(\sum_{i=u\ell_s+1}^j ((\pi_s h)^2(X_i))^*\right)^{1/2} \leq F_u(j).$

The final set, $\Omega_{3,u}$ is very close in nature to $\Omega_{2,u}$. It is needed to control the coordinates of “very small” increments – when $s < s_0$, if such an integer exists.

Definition 4.7 *If $\eta_0 < \log(eN)$, let $\Omega_{3,u}$ be the event on which for every $h \in H$, every $0 \leq s < s_0$ and $1 \leq j \leq N$,*

$$\left(\sum_{i=1}^j ((\Delta_s h)^2(X_i))^*\right)^{1/2} \leq f_u(\Delta_s h, j), \quad \left(\sum_{i=1}^j ((\pi_{s_0} h)^2(X_i))^*\right)^{1/2} \leq F_u(j).$$

If $\eta_0 \geq \log(eN)$ set $\Omega_{3,u} = \Omega^N$.

It turns out that on the event $\Omega_{1,u} \cap \Omega_{2,u} \cap \Omega_{3,u}$, the set $P_\sigma H$ is indeed well behaved. Let

$$\gamma_u = \inf \sup_{h \in H} \sum_{s > s_0} \theta_{u,s}(\Delta_s h), \quad (4.1)$$

with the infimum is taken with respect to all (η_s) -admissible sequences. From here on we will assume that $(H_s)_{s \geq 0}$ is an almost optimal $(\eta_s)_{s \geq 0}$ -admissible sequence.

Lemma 4.8 *There exists absolute constants c_1 and c_2 for which the following holds. Let $(\theta_{u,s})_{s \geq s_0}$ be functionals, and for $s < s_0$ set $\theta_{u,s} = 0$. For every $u \geq c_1$, on the event $\Omega_{1,u} \cap \Omega_{2,u} \cap \Omega_{3,u}$, for every $h \in H$ and $I \subset \{1, \dots, N\}$,*

1. *if $\eta_s \leq N$ then*

$$\left(\sum_{i \in I} (\Delta_s h)^2(X_i)\right)^{1/2} \leq \theta_{u,s}(\Delta_s h) + f_u(\Delta_s h, |I|),$$

and if $\eta_s > N$,

$$\left(\sum_{i \in I} (\Delta_s h)^2(X_i)\right)^{1/2} \leq \theta_{u,s}(\Delta_s h).$$

2.

$$\left(\sum_{i \in I} h^2(X_i) \right)^{1/2} \leq \gamma_u + \sum_{\{i: 2^i \leq |I|\}} F(c_2 u 2^i) + R_{s_0}(h, I),$$

where $R_{s_0, I}(h) = \theta_{u, 0}(\pi_0 h)$ if $s_0 = 0$ and $R_{s_0}(h, I) = \min\{\theta_{u, s_0}(\pi_{s_0} h), F_u(|I|)\}$ otherwise.

Proof. First, assume that $\log(eN) \leq \eta_s \leq N$ (i.e. $s \geq s_0$) and recall that ℓ_s is the largest integer for which $\eta_s \geq \ell \log(eN/\ell)$. If $|I| \leq u\ell_s$ then the claim follows from the definition of $\theta_{u, s}$ and the set $\Omega_{1, u}$. If $|I| \geq u\ell_s$, then

$$\left(\sum_{i \in I} (\Delta_s h)^2(X_i) \right)^{1/2} \leq \left(\sum_{i=1}^{u\ell_s} ((\Delta_s h)^2(X_i))^* \right)^{1/2} + \left(\sum_{i=u\ell_s+1}^{|I|} ((\Delta_s h)^2(X_i))^* \right)^{1/2},$$

and the claim is evident from the definition of the function f_u and the set $\Omega_{2, u}$.

If, on the other hand, $\eta_s < \log(eN)$ then $s_0 > 0$ and the assertion follows from the definition of $\Omega_{3, u}$.

The second part of (1) follows from the definition of $\Omega_{1, u}$.

Turning to (2), we shall treat two cases. First, consider the case $|I| \geq u\ell_{s_0}$ and observe that it suffices to estimate $(\sum_{i=1}^{u\ell_{s+1}} ((\pi_s h)^2(X_i))^*)^{1/2}$. Indeed, let s be an integer for which $u\ell_s \leq |I| < u\ell_{s+1}$. Since ℓ_{s+1} is nondecreasing, then on $\Omega_{1, u}$,

$$\begin{aligned} \left(\sum_{i=1}^{u\ell_{s+1}} (h^2(X_i))^* \right)^{1/2} &\leq \sum_{j \geq s+1} \left(\sum_{i=1}^{u\ell_{s+1}} ((\Delta_j h)^2(X_i))^* \right)^{1/2} + \left(\sum_{i=1}^{u\ell_{s+1}} ((\pi_s h)^2(X_i))^* \right)^{1/2} \\ &\leq \sum_{j \geq s+1} \theta_{u, j}(\Delta_j h) + \left(\sum_{i=1}^{u\ell_{s+1}} ((\pi_s h)^2(X_i))^* \right)^{1/2}. \end{aligned}$$

If $J \subset I$ is the set of the largest $u\ell_s$ coordinates of $((\pi_s h)(X_i))_{i=1}^N$ in I , then the coordinate projections satisfy that

$$P_I((\pi_s h)(X_i))_{i=1}^N = P_J((\pi_{s-1} h)(X_i))_{i=1}^N + P_J((\Delta_s h)(X_i))_{i=1}^N + P_{I \setminus J}((\pi_s h)(X_i))_{i=1}^N,$$

and thus,

$$\begin{aligned}
& \left(\sum_{i \in I} (\pi_s h)^2(X_i) \right)^{1/2} \leq \left(\sum_{i=1}^{u\ell_{s+1}} ((\pi_s h)^2(X_i))^* \right)^{1/2} \\
& \leq \max_{|I_1|=u\ell_s} \left(\sum_{i \in I_1} (\pi_{s-1} h)^2(X_i) \right)^{1/2} + \max_{|I_1|=u\ell_s} \left(\sum_{i \in I_1} (\Delta_s h)^2(X_i) \right)^{1/2} \\
& + \left(\sum_{i=u\ell_s+1}^{u\ell_{s+1}} ((\pi_s h)^2(X_i))^* \right)^{1/2}.
\end{aligned}$$

Hence, if we set $\mathcal{U}_{j,s}(h) = \max_{|I|=u\ell_{j+1}} \left(\sum_{i \in I} (\pi_s h)^2(X_i) \right)^{1/2}$ then for every s , and every $h \in H$

$$\begin{aligned}
\mathcal{U}_{s,s}(h) & \leq \mathcal{U}_{s-1,s-1}(h) + \max_{|I_1|=u\ell_s} \left(\sum_{i \in I_1} (\Delta_s h)^2(X_i) \right)^{1/2} \\
& + \left(\sum_{i=u\ell_s+1}^{u\ell_{s+1}} ((\pi_s h)^2(X_i))^* \right)^{1/2} \\
& \leq \mathcal{U}_{s-1,s-1}(h) + \theta_{u,s}(\Delta_s h) + F_u(u\ell_{s+1}).
\end{aligned}$$

Summing over all $s > s_0$,

$$\mathcal{U}_{s,s}(h) \leq \sum_{j=s_0+1}^s \theta_{u,j}(\Delta_j h) + \sum_{j=s_0+1}^{s+1} F_u(u\ell_j) + \mathcal{U}_{s_0,s_0}(h),$$

and thus, for every $h \in H$ and every $I \subset \{1, \dots, N\}$,

$$\left(\sum_{i \in I} h^2(X_i) \right)^{1/2} \leq \sum_{s>s_0} \theta_{u,s}(\Delta_s h) + \sum_{\{s>s_0: \ell_s \leq |I|\}} F_u(u\ell_{s+1}) + \mathcal{U}_{s_0,s_0}(h).$$

Next, one has to bound $\sup_{h \in H} \max_{|I| \leq u\ell_{s_0+1}} \left(\sum_{i \in I} (\pi_{s_0} h)^2(X_i) \right)^{1/2}$. This is at most $\theta_{u,s_0}(\pi_{s_0} h)$ on $\Omega_{1,u}$ and when $s_0 > 0$, it is also bounded by $F_u(u\ell_{s_0}) \leq F_u(|I|)$ on $\Omega_{3,u}$.

The claim in this case follows since ℓ_s grows exponentially for $s \geq s_0$, and thus

$$\sum_{\{s \geq s_0: \ell_s \leq |I|\}} F_u(u\ell_{s+1}) \leq \sum_{\{i: 2^i \leq |I|\}} F_u(cu2^i)$$

for a suitable absolute constant c .

Turning to the second case, if $|I| \leq u\ell_{s_0}$, note that

$$\begin{aligned} \left(\sum_{i \in I} h^2(X_i) \right)^{1/2} &\leq \sum_{s > s_0} \left(\sum_{i \in I} (\Delta_s h)^2(X_i) \right)^{1/2} + \left(\sum_{i \in I} (\pi_{s_0} h)^2(X_i) \right)^{1/2} \\ &\leq \sum_{s > s_0} \theta_{u,s}(\Delta_s h) + \min\{\theta_{u,s_0}(\pi_{s_0} h), F_u(|I|)\}. \end{aligned}$$

■

For Lemma 4.8 to have any meaning, one has to identify the functionals f_u , F_u and $\theta_{u,s}$ in the cases one is interested in. Our next goal is to study the functions f_u and F_u under various tail assumptions on functions in H , and naturally, the two families of tail estimates we will be interested in are when H has a bounded diameter in L_{ψ_β} or in L_q for $q > 2$.

If $H \subset L_{\psi_\beta}$, then for every $h \in H$, $Pr(|h| \geq y) \leq \exp(-(y/\|h\|_{\psi_\beta})^\beta)$. Thus, for $\varepsilon \geq 1$ and every j ,

$$y_j \lesssim \varepsilon \|h\|_{\psi_\beta} \log^{1/\beta}(eN/j), \quad z_j \lesssim \varepsilon \sup_{h \in H} \|h\|_{\psi_\beta} \log^{1/\beta}(eN/j).$$

Hence, if $d_{\psi_\beta} = \sup_{h \in H} \|h\|_{\psi_\beta}$, then

$$\begin{aligned} F_u(i) &\lesssim \varepsilon \sqrt{u} \left(\sum_{j=1}^{\log_2 i} 2^j z_{2^j}^2 \right)^{1/2} \lesssim_\beta \varepsilon \sqrt{u} d_{\psi_\beta} \left(\sum_{j=1}^{\log_2 i} 2^j \log^{2/\beta}(eN/2^j) \right)^{1/2} \\ &\lesssim_\beta \varepsilon \sqrt{u} d_{\psi_\beta} \sqrt{i} \log^{1/\beta}(eN/i) \sim_\beta \varepsilon \sqrt{u} d_{\psi_\beta} \phi_\beta(i), \end{aligned}$$

and in a similar fashion,

$$f_u(h, i) \lesssim_\beta \varepsilon \sqrt{u} \|h\|_{\psi_\beta} \sqrt{i} \log^{1/\beta}(eN/i) \sim_\beta \varepsilon \sqrt{u} \|h\|_{\psi_\beta} \phi_\beta(i).$$

Using the same argument, if $h \in L_q$ then $Pr(|h| \geq \|h\|_{L_q} y) \leq 1/y^q$ and for any $0 < \varepsilon < q/2 - 1$, $y_j = \|h\|_{L_q} (N/2^j)^{(1+\varepsilon)/q}$. If $\sup_{h \in H} \|h\|_{L_q} = d_{L_q}$, $q > 2$ and $c_{q,\varepsilon} = 1 - 2(1+\varepsilon)/q$ then

$$\begin{aligned} F_u(i) &\lesssim \sqrt{u} d_{L_q} \left(\sum_{j=1}^{\log_2 i} 2^j (N/2^j)^{2(1+\varepsilon)/q} \right)^{1/2} \lesssim c_{q,\varepsilon}^{-1} \sqrt{u} d_{L_q} \sqrt{i} \left(\frac{N}{i} \right)^{(1+\varepsilon)/q} \\ &\sim c_{q,\varepsilon}^{-1} \sqrt{u} d_{L_q} \phi_{q,\varepsilon}(i), \end{aligned}$$

and

$$f_u(h, i) \lesssim c_{q,\varepsilon}^{-1} \sqrt{u} \|h\|_{L_q} \sqrt{i} \left(\frac{N}{i} \right)^{(1+\varepsilon)/q} \sim c_{q,\varepsilon}^{-1} \sqrt{u} \|h\|_{L_q} \phi_{q,\varepsilon}(i).$$

Combining these observations with the estimates of Lemma 4.8 and noting that if $s_0 > 0$ then $R_{s_0, I}(h) \lesssim \sqrt{u} d_{L_q} \phi_{q,\varepsilon}(|I|)$, one reaches the following corollary.

Corollary 4.9 *Let $(\theta_{u,s})_{s \geq s_0}$ be a sequence of functionals and for $s < s_0$ let $\theta_{u,s} = 0$. If H is bounded in L_q for $q > 2$, then on $\Omega_{1,u} \cap \Omega_{2,u} \cap \Omega_{3,u}$, for every $h \in H$ and every $I \subset \{1, \dots, N\}$*

1. *if $\eta_s \leq N$,*

$$\left(\sum_{i \in I} (\Delta_s h)^2(X_i) \right)^{1/2} \lesssim \theta_{u,s}(\Delta_s h) + c_{q,\varepsilon}^{-1} \sqrt{u} \|\Delta_s h\|_{L_q} \phi_{q,\varepsilon}(|I|),$$

and if $\eta_s > N$ then

$$\left(\sum_{i \in I} (\Delta_s h)^2(X_i) \right)^{1/2} \lesssim \theta_{u,s}(\Delta_s h).$$

2.

$$\left(\sum_{i \in I} h^2(X_i) \right)^{1/2} \lesssim \sum_{s \geq 0} \theta_{u,s}(\Delta_s h) + c_{q,\varepsilon}^{-1} \sqrt{u} d_{L_q} \phi_{q,\varepsilon}(|I|).$$

A similar bound holds when H is bounded in L_{ψ_β} .

5 Estimates on $\Omega_{i,u}$ and the choice of functionals

We will begin by showing that $\Omega_{2,u}$ is a large set, almost regardless of any assumptions on ϕ , an observation that is based on the same idea as Lemma 4.4.

Lemma 5.1 *There exist absolute constants c_1 and c_2 such that, for every $\varepsilon > 0$ and $u \geq c_1/\varepsilon$, $\Pr(\Omega_{2,u}) \geq 1 - 2 \exp(-c_2 \varepsilon u \eta_{s_0})$.*

Proof. Recall that by Lemma 4.4, for any random variable Y , with probability at least $1 - 2 \exp(-c_1 u \varepsilon \ell \log(eN/\ell))$, for every integer $k > u\ell$,

$$\sum_{i=u\ell+1}^k (Y_i^2)^* \leq f_u(Y, k). \quad (5.1)$$

Let $\ell = \ell_{s_0}$, and since $\eta_s \sim \ell_s \log(eN/\ell_s)$ and $|\Delta_s H| \lesssim 2^{\eta_s}$, then for $u \geq c_3/\varepsilon$, (5.1) holds uniformly for every $h \in \Delta_s H$ with probability at least $1 - 2\exp(-c_4 u \varepsilon \eta_s)$. The analogous claim holds for functions in H_s as well, with the uniform bound of F_u replacing f_u . Summing over all $s \geq s_0$ and since (η_s) grows exponentially, the claim follows. \blacksquare

Since $\Omega_{2,u}$ is always large, and since $\Omega_{3,u}$ will behave in a very similar way when $s_0 > 0$, the crucial point in the construction of a good decomposition of $P_\sigma H$ is a correct choice of $\theta_{u,s}$ and estimates on $\Omega_{1,u}$.

The functionals $\theta_{u,s}$ capture the geometry of H , and thus have to be selected according to the information one has on the class. We will present two examples of such choices, each leading to one of our two main results. The first one will be based on “global” structure like metric entropy, while the second uses accurate estimates on each “chain”.

5.1 The ball B_2^n – global estimates

Let μ be an unconditional measure on \mathbb{R}^n , set $H = \{\langle t, \cdot \rangle : t \in B_2^n\}$ to be a class of linear functionals on (\mathbb{R}^n, μ) – and from here on we will identify the class $\{\langle t, \cdot \rangle : t \in T\}$ with its indexing set T . We will also assume that μ satisfies the p -small diameter, L_q moment assumption for some $p > 2$ and $q > 2$; that is, μ is supported in $\kappa_1 n^{1/p} B_p^n$, and for every $x \in \mathbb{R}^n$, $\|\langle x, \cdot \rangle\|_{L_q} \leq \kappa_2 \|x\|_{\ell_2^n}$.

Let $\kappa_4 \geq 10$ be an absolute constant to be fixed later, set $2^{s_1} \sim n^\delta$ for $\delta < 1/2 - 1/2(p-1)$, and put

$$\eta_s = \kappa_4 2^{s+s_1} \max\{\log(en/2^{s+s_1}), 1\}.$$

Note that $s_0 = 0$ as long as $\eta_0 \sim 2^{s_1} \log(en/2^{s_1}) \geq \log(eN)$, i.e., if $n^\delta \log(n) \gtrsim \log(eN)$ – which we will assume is the case, since our main interest is when $N \sim n$.

If $X = (x_1, \dots, x_n)$ is distributed according to μ then for every $1 \leq \ell \leq n$, set $M_\ell = \|(\sum_{i=1}^\ell (x_i^2)^*)^{1/2}\|_{L_\infty}$. Define the following functionals (which, in this case, will be constants depending only on u and s): let $\theta_{u,0} = c\sqrt{u}\eta_0^{1/2} n^{1/p} 2^{(s+s_1)(1/2-1/p)}$, if $2^{s+s_1} \leq n$, set $\theta_{u,s} = c\sqrt{u}\eta_s^{1/2} n^{1/p} 2^{-(s+s_1)/p}$ and if $\eta_s \geq n$ put $\theta_{u,s} = c\sqrt{u}\eta_s^{1/2} 2^{-2^s/n}$, where $c = c(\kappa_1, p, \delta)$.

Theorem 5.2 *For every κ_1 , $p > 2$ and $\delta < 1/2 - 1/2(p-1)$ there exist constants c_1, c_2 and c_3 that depend only on κ_1, p and δ for which the following holds. There is an $(\eta_s)_{s \geq 0}$ -admissible sequence of B_2^n , for which, if $u \geq c_1$,*

then $Pr(\Omega_{1,u}) \geq 1 - \exp(-c_2 n^\delta)$ and

$$\sup_{t \in B_2^n} \sum_{s \geq 0} \theta_{u,s}(\langle \Delta_s t, \cdot \rangle) \leq c_3 \sqrt{u} \sqrt{n}.$$

Observe that by the p -small diameter assumption, $M_\ell \lesssim_p n^{1/p} \ell^{1/2-1/p}$. Also, since μ is unconditional, then for every $I \subset \{1, \dots, n\}$ and v supported on I ,

$$\|\langle v, \cdot \rangle\|_{\psi_2} \lesssim \|v\|_{\ell_\infty^I} M_{|I|}. \quad (5.2)$$

Indeed, by the unconditionality of μ , (x_1, \dots, x_n) has the same distribution as $(\varepsilon_1 x_1, \dots, \varepsilon_n x_n)$. Hence, for every $r \geq 1$

$$\begin{aligned} \|\langle v, \cdot \rangle\|_{L_r} &\sim (\mathbb{E}_X \mathbb{E}_\varepsilon |\sum_{i \in I} \varepsilon_i x_i v_i|^r)^{1/r} \lesssim \left(\mathbb{E}_X r^{r/2} (\sum_{i \in I} v_i^2 x_i^2)^{r/2} \right)^{1/r} \\ &\lesssim \sqrt{r} \|v\|_{\ell_\infty^I} M_{|I|}. \end{aligned}$$

We will also need a few ψ_2 entropy estimates. Set $B_{\psi_2} = \{v \in \mathbb{R}^n : \|\langle v, \cdot \rangle\|_{\psi_2} \leq 1\}$, and for $K, L \subset \mathbb{R}^n$ denote by $N(K, L)$ the minimal number of translates of L needed to cover K .

Lemma 5.3 *If $I \subset \{1, \dots, n\}$ then for every $\varepsilon > 0$, $\log N(B_2^I, \varepsilon B_{\psi_2}) \lesssim M_{|I|}^2 / \varepsilon^2$. Moreover, for $\varepsilon \leq 1$, $\log N(B_2^n, \varepsilon B_{\psi_2}) \lesssim n \log(2/\varepsilon)$.*

Proof. By the dual Sudakov inequality (see, e.g. [20]), if $B_{\|\cdot\|}$ is a unit ball of a norm on \mathbb{R}^I and $G = (g_i)_{i \in I}$ is a standard Gaussian vector on \mathbb{R}^I , then $\log N(B_2^n, \varepsilon B_{\|\cdot\|}) \lesssim (\mathbb{E} \|G\|)^2 / \varepsilon^2$. Since $\|f\|_{\psi_2} \leq \mathbb{E} \exp(f^2)$ and $(\sum_{i \in I} x_i^2)^{1/2} \leq M_{|I|}$ almost surely, then by changing the order of integration,

$$\mathbb{E} \|G/cM_{|I|}\|_{\psi_2} \leq \mathbb{E}_X \mathbb{E}_G (\exp((\sum_{i \in I} g_i x_i)^2 / c^2 M_{|I|}^2) | X) \leq 2$$

for a suitable absolute constant c , proving the first part.

For the second part, note that $N(B_2^n, \varepsilon B_{\psi_2}) \leq N(B_2^n, B_{\psi_2}) \cdot N(B_{\psi_2}, \varepsilon B_{\psi_2})$. By the first part, $\log N(B_2^n, B_{\psi_2}) \lesssim n$, while a standard volumetric estimate shows that $N(B_{\psi_2}, \varepsilon B_{\psi_2}) \leq (5/\varepsilon)^n$. \blacksquare

Next, let us define the sets T_s . If $2^{s+s_1} > n$, let T_s be a maximal ε_s separated subset of B_2^n relative to the ψ_2 norm and of cardinality 2^{η_s} . If $2^{s+s_1} \leq n$, let T_s be a maximal ε_s separated subset of $U_{2^{s+s_1}} = \{x \in B_2^n : |\text{supp}(x)| \leq 2^{s+s_1}\}$ with respect to the ψ_2 norm, and of cardinality 2^{η_s} . Given a vector $t \in B_2^n$, we will define the functions π_s as follows. If $2^{s+s_1} > n$,

$\pi_s t$ is a best ψ_2 approximation of t in T_s . For $2^{s+s_1} \leq n$ one combines approximation and dimension reduction. Set s_* to satisfy that $2^{s_*+s_1} = n$ (and without loss of generality we will assume that such an integer exists). If $v = \pi_{s_*} t$, let $I_{n/2}$ be the set of the largest $n/2$ coordinates of v , and put $\pi_{s_*-1} t$ to be the best approximation of the coordinate projection $P_{I_{n/2}} v$ in T_{s_*-1} , and so on.

Lemma 5.4 *There exists an absolute constant c such that for every $t \in B_2^n$, if $s > s_*$ (i.e., if $\eta_s > \kappa_3 n$), then $\|\langle \Delta_s t, \cdot \rangle\|_{\psi_2} \leq c 2^{-2^{s+s_1}/n}$, and if $0 < s \leq s_*$ then $\|\langle \Delta_s t, \cdot \rangle\|_{\psi_2} \leq c 2^{-(s+s_1)/2} M_{2^{s+s_1}}$.*

Proof. First consider $s > s_*$. Note that $\|\langle \Delta_s t, \cdot \rangle\|_{\psi_2} \leq \|\langle t - \pi_s t, \cdot \rangle\|_{\psi_2} + \|\langle t - \pi_{s-1} t, \cdot \rangle\|_{\psi_2} \leq \varepsilon_s + \varepsilon_{s-1}$, and by the covering numbers estimate from Lemma 5.3, in that range $\varepsilon_s \lesssim 2^{-2^{s+s_1}/n}$.

In the range $s \leq s_*$, $\Delta_s t = u + w$, where w consists of the smallest 2^{s+s_1-1} coordinates of $\pi_s t \in B_2^I$ for some $|I| = 2^{s+s_1}$, and u is an ε_{s-1} -approximation of the largest 2^{s+s_1-1} coordinates of $\pi_s t$. Therefore, $\|\langle \Delta_s t, \cdot \rangle\|_{\psi_2} \leq \|\langle w, \cdot \rangle\|_{\psi_2} + \varepsilon_{s-1}$. Recall that for every such s , $U_{2^{s+s_1}}$ is a union of $\binom{n}{2^{s+s_1}}$ balls of dimension 2^{s+s_1} , then

$$\begin{aligned} \log N(U_{2^{s+s_1}}, \varepsilon B_{\psi_2}) &\leq 2^{s+s_1} \log(en/2^{s+s_1}) + \max_{|J|=2^{s+s_1}} \log N(B_2^J, \varepsilon B_{\psi_2}) \\ &\lesssim 2^{s+s_1} \log(en/2^{s+s_1}) + M_{2^{s+s_1}}^2 / \varepsilon^2. \end{aligned}$$

Note that for a suitable choice of κ_4 , $\log |T_s| \geq 2 \cdot 2^{s+s_1} \log(en/2^{s+s_1})$. Therefore, $\varepsilon_s \leq 2^{-(s+s_1)/2} M_{2^{s+s_1}}$, and applying (5.2), $\|\langle w, \cdot \rangle\|_{\psi_2} \lesssim \|w\|_{\ell_\infty^I} M_{|I|} \lesssim 2^{-(s+s_1)/2} M_{2^{s+s_1}}$. \blacksquare

Proof of Theorem 5.2. Observe that $\|Y\|_{\psi_2}^2 = \|Y^2\|_{\psi_1}$, and thus, by a standard application of Bernstein's inequality, for every integer m ,

$$Pr \left(\sum_{i=1}^m Y_i^2 \geq m \|Y\|_{\psi_2}^2 t^2 \right) \leq 2 \exp(-cm \min\{t^2, t^4\}).$$

Therefore, if w is large enough, then

$$\begin{aligned} &Pr \left(\sum_{i=1}^{u\ell_s} (Y_i^2)^* \geq w^2 \|Y\|_{\psi_2}^2 \cdot u\ell_s \log(eN/u\ell_s) \right) \\ &\leq \binom{N}{u\ell_s} \cdot 2 \exp(-cw^2 u\ell_s \log(eN/u\ell_s)) \leq 2 \exp(-c_1 w^2 u\ell_s \log(eN/u\ell_s)). \end{aligned}$$

Moreover, $u\ell_s \log(en/u\ell_s) \lesssim u\eta_s$, and for $u \geq 1$, $u\ell_s \log(en/u\ell_s) \gtrsim \eta_s$, implying that with probability at least $1 - 2\exp(-c_2 w^2 \eta_s)$,

$$\left(\sum_{i=1}^{u\ell_s} (Y_i^2)^* \right)^{1/2} \lesssim w u^{1/2} \|Y\|_{\psi_2} \eta_s^{1/2}.$$

Also, with probability at least $1 - 2\exp(-c_2 w^2 \eta_s)$, if $\eta_s \geq N$ then

$$\left(\sum_{i=1}^N (Y_i^2)^* \right)^{1/2} \lesssim w u^{1/2} \|Y\|_{\psi_2} \eta_s^{1/2}.$$

Using Lemma 5.4 and summing the probability estimates, it is evident that with probability at least $1 - 2\exp(-c_3 w^2 \eta_0)$, the following holds: if $\eta_s \geq N$ then

$$\sup_{t \in B_2^n} \left(\sum_{i=1}^N (\langle \Delta_s t, X_i \rangle^2)^* \right)^{1/2} \lesssim w u^{1/2} \eta_s^{1/2} 2^{-2^{s+s_1}/n},$$

if $\kappa_4 n \leq \eta_s < N$, then

$$\sup_{t \in B_2^n} \left(\sum_{i=1}^{u\ell_s} (\langle \Delta_s t, X_i \rangle^2)^* \right)^{1/2} \lesssim w u^{1/2} \eta_s^{1/2} 2^{-2^{s+s_1}/n},$$

and if $s > 0$ and $\eta_s \leq \kappa_4 n$ then

$$\sup_{t \in B_2^n} \left(\sum_{i=1}^{u\ell_s} (\langle \Delta_s t, X_i \rangle^2)^* \right)^{1/2} \lesssim w u^{1/2} \eta_s^{1/2} 2^{-(s+s_1)/2} M_{2^{s+s_1}}.$$

Finally, since $\eta_0 = \kappa_4 2^{s_1} \log(en/2^{s_1})$ then $\ell_0 \lesssim 2^{s_1}$. Moreover, $|\text{supp}(\pi_0 t)| \leq 2^{s_1}$ and by (5.2), $\|\langle \pi_0 t, \cdot \rangle\|_{\psi_2} \leq M_{2^{s_1}}$. Hence, with probability at least $1 - 2\exp(-c_4 w^2 \eta_0)$,

$$\sup_{t \in B_2^n} \left(\sum_{i=1}^{u\ell_s} (\langle \pi_0 t, X_i \rangle^2)^* \right)^{1/2} \lesssim w u^{1/2} \eta_0^{1/2} M_{2^{s_1}}.$$

Since $M_\ell \lesssim_p n^{1/p} \ell^{1/2-1/p}$, then $Pr(\Omega_{1,u}) \geq 1 - 2\exp(-c_5 \eta_0) = 1 - 2\exp(-c_5 2^{s_1})$ for the desired functionals $\theta_{u,s}$. It remains to choose s_1 and estimate $\sum_{s \geq 0} \theta_{u,s}$.

Note that if $2^{s_1} \sim n^\delta$ for $\delta < 1/2 - 1/2(p-1)$, then

$$\theta_0 \sim \eta_0^{1/2} M_{2^{s_1}} \sim_{\kappa_1, p} 2^{s_1/2} \log^{1/2}(en/2^{s_1}) n^{1/p} 2^{s_1(1/2-1/p)} \leq c_6(\kappa_1, p, \delta) \sqrt{n}. \quad (5.3)$$

Also,

$$\begin{aligned}
\sum_{\{s>0:\eta_s\leq\kappa_3n\}} \theta_s &\lesssim \sum_{\{s:\eta_s\leq\kappa_3n\}} \eta_s^{1/2} 2^{-(s+s_1)/2} M_{2^{s+s_1}} \\
&\lesssim_{\kappa_1,p,\delta} n^{1/p} \sum_{\{s:2^{s+s_1}\leq n\}} 2^{(s_1+s)(1/2-1/p)} \log^{1/2}(en/2^{s+s_1}) \leq c_6(\kappa_1,p,\delta)\sqrt{n},
\end{aligned} \tag{5.4}$$

and

$$\sum_{\{s:\eta_s>\kappa_3n\}} \theta_s \lesssim_{\kappa_1,\kappa_3,p} \sum_{\{s:\eta_s>\kappa_3n\}} 2^{(s+s_1)/2} 2^{-2^{s+s_1}/n} \leq c_6(\kappa_1,p,\delta)\sqrt{n}. \tag{5.5}$$

■

Corollary 5.5 *There exist absolute constants c_1, c_2 and c_3 and c_4 that depend on $\kappa_1, \kappa_2, p, \delta$, for which the following holds. If μ is as above and $\varepsilon > 0$, then B_2^n has an $(\eta_s)_{s\geq 0}$ -admissible sequence $(T_s)_{s\geq 0}$ for which, for $u \geq c_1/\varepsilon$, with probability at least $1 - 2\exp(-c_2\varepsilon un) - 2\exp(-c_3n^\delta)$, for every $t \in B_2^n$ and every $I \subset \{1, \dots, N\}$,*

1. if $\eta_s \leq N$,

$$\left(\sum_{i \in I} (\langle \Delta_s t, X_i \rangle)^2 \right)^{1/2} \leq c_4 \theta_{u,s} + c_{q,\varepsilon}^{-1} \sqrt{u} \|\Delta_s t\|_{\ell_2^n} \phi_{q,\varepsilon}(|I|),$$

and if $\eta_s > N$ then

$$\left(\sum_{i \in I} (\langle \Delta_s t, X_i \rangle)^2 \right)^{1/2} \leq c_4 \theta_{u,s}.$$

2.

$$\begin{aligned}
\left(\sum_{i \in I} (\langle t, X_i \rangle)^2 \right)^{1/2} &\leq c_4 \sum_s \theta_{u,s} + c_{q,\varepsilon}^{-1} \sqrt{u} \phi_{q,\varepsilon}(|I|) \\
&\lesssim \sqrt{u} \sqrt{n} + c_{q,\varepsilon}^{-1} \sqrt{u} \phi_{q,\varepsilon}(|I|).
\end{aligned}$$

We will separate our treatment to the cases $q > 4$ and $2 < q \leq 4$. First, if $q > 4$, let $\varepsilon = (q/4 - 1)/2$ and note that $c_{q,\varepsilon} \geq 1/2$. Also, since $\|t\|_{\ell_2^n} \sim_{\kappa_2}$

$\|\langle t, \cdot \rangle\|_{L_q} \lesssim_{\kappa_2} \|\langle t, \cdot \rangle\|_{\psi_2}$, then by the same computation as in (5.3), (5.4) and (5.5),

$$B_4 = \sup_{t \in B_2^n} \sum_{s \geq 0} \eta_s^{1/2} \|\Delta_s t\|_{L_q} \lesssim_{\kappa_1, \kappa_2, p, \delta} \sqrt{n}.$$

We thus have:

Theorem 5.6 *For every $\kappa_1, \kappa_2, q > 4, p > 2$ and $\delta < 1/2 - 1/2(p-1)$, there exist constants c_0, c_1, c_2 and c_3 which depend on κ_1, κ_2, p, q and δ , and an absolute constant c_4 for which the following holds. If μ is as above, and $N \leq \exp(c_0 n^\delta)$, then for every $u \geq c_1$, with μ^N -probability at least $1 - 2 \exp(-c_2 n^\delta)$, $P_\sigma(B_2^n)$ satisfies that*

$$\sup_{t \in B_2^n} \left| \frac{1}{N} \sum_{i=1}^N \varepsilon_i \langle X_i, t \rangle^2 \right| \leq c_3 r u \left(\sqrt{\frac{n}{N}} + \frac{n}{N} \right),$$

with probability at least $1 - 2 \exp(-c_4 n r^2)$ relative to the Bernoulli random variables.

Turning to the case $2 < q \leq 4$, recall that for $0 < \varepsilon < q/2 - 1$, $B_{q,\varepsilon} = \sum_{\{s: \eta_s \leq N\}} \eta_s^{1-2(1+\varepsilon)/q} \|\Delta_s v\|_{L_q}$. Assume that μ is as above and satisfies the p -small diameter assumption for $p > q/(q/2 - 1)$. Then, for $0 < \varepsilon < q/2 - 1 - q/p$ (i.e. if $1 - (2(1+\varepsilon)/q) - 1/p > 0$),

$$\begin{aligned} B_{q,\varepsilon} &\lesssim \sum_{\{s: 2^{s+s_1} \leq n\}} (2^{s+s_1} \log(en/2^{s+s_1}))^{1-2(1+\varepsilon)/q} 2^{-(s+s_1)/2} n^{1/p} 2^{(s+s_1)(1/2-1/p)} \\ &+ \sum_{\{s: 2^{s+s_1} > n\}} 2^{(s+s_1)(1-2(1+\varepsilon)/q)} 2^{-2(s+s_1)/n} \lesssim \frac{n^{1-2(1+\varepsilon)/q}}{1 - 2(1+\varepsilon)/q - 1/p}. \end{aligned}$$

Therefore, one has

Theorem 5.7 *Let $2 < q \leq 4, p > (1 - 2/q)^{-1}$ and $0 < \varepsilon < 2/q - 1 - q/p$. If μ and δ are as above, $u \gtrsim 1/\varepsilon$, and $N \lesssim \exp(c_0 n^\delta)$, then with μ^N probability at least $1 - 2 \exp(-c_1 \varepsilon u n) - 2 \exp(-c_2 n^\delta)$, $P_\sigma(B_2^n)$ satisfies that*

$$\sup_{t \in B_2^n} \left| \frac{1}{N} \sum_{i=1}^N \varepsilon_i \langle X_i, t \rangle^2 \right| \lesssim_{\kappa_1, \kappa_2, p, \delta} \frac{r u}{(1 - 2(1+\varepsilon)/q)^2} \left(\left(\frac{n}{N} \right)^{1-(2/q)-2\varepsilon/q} + \sqrt{\frac{n}{N}} + \frac{n}{N} \right).$$

with probability at least $1 - 2 \exp(-c_3 n r^2)$ relative to the Bernoulli random variables.

In particular, taking $\varepsilon \sim 1/\log(eN/n)$, then for every such N satisfying that $N \gtrsim_{\kappa_1, \kappa_2, q, p} n$, and any $u \geq_{\kappa_1, \kappa_2, q, p} \log(eN/n)$,

$$\sup_{t \in B_2^n} \left| \frac{1}{N} \sum_{i=1}^N \varepsilon_i \langle X_i, t \rangle^2 \right| \lesssim_{\kappa_1, \kappa_2, q, p} r u \left(\frac{n}{N} \right)^{1-2/q}.$$

5.2 Unconditional log-concave measures

We will now present a different way of bounding $\Omega_{1,u}$ (and $\Omega_{3,u}$ if needed) by estimating the moments of the increments $\Delta_s h$, and selecting the functionals $\theta_{u,s}$ accordingly.

For every $s \geq 0$ and $h \in H$, set

$$Z_s^2(h) = \sum_{i=1}^{\min\{u\ell_{s+1}, N\}} ((\Delta_s h)^2(X_i))^*, \quad Z_{s_0}^2(h) = \sum_{i=1}^{\min\{u\ell_{s_0+1}, N\}} ((\pi_{s_0} h)^2(X_i))^*.$$

In light of Theorem B, we will assume that H is a bounded subset of L_{ψ_1} (although what we do here can be extended to other moment assumptions), and thus one may control $\Omega_{2,u}$ using ϕ_β for $\beta = 1$ and ε which will be selected later.

Lemma 5.8 *There exist absolute constants c_1, c_2 and c_3 for which the following holds. For $u \geq c_1$, with probability at least $1 - 2\exp(-c_1 u \eta_{s_0})$, for every $s \geq s_0$ and every $h \in H$, $Z_s(h) \leq e \|Z_s(h)\|_{L_{2u\eta_{s+1}}}$.*

Proof. If Z is a nonnegative random variable then $\Pr(Z \geq e \|Z\|_{L_q}) \leq \exp(-q)$. Thus, for a fixed s and every $h \in H$, $Z_s(h) \leq e \|Z_s(h)\|_{L_{2u\eta_{s+1}}}$ with probability at least $1 - \exp(-2u\eta_{s+1})$. Since $\log |\Delta_s H| \lesssim \eta_{s+1}$ and because there are at most $\exp(u\ell_{s+1} \log(eN/u\ell_{s+1})) \leq \exp(u\eta_{s+1})$ subsets of $\{1, \dots, N\}$ of cardinality $u\ell_{s+1}$, the same probability estimate holds uniformly for every $h \in H$ (with a different constant). Summing the probabilities for every $s \geq s_0$ and repeating the same argument for H_{s_0} concludes the proof. \blacksquare

Next, one has to control the moments appearing in Lemma 5.8, which is based on the following result, due to Latała [17].

Theorem 5.9 *Let X_1, \dots, X_m be independent, distributed according to a nonnegative random variable X . Then for every $p \geq 1$,*

$$\left\| \sum_{i=1}^m X_i \right\|_{L_p} \sim \left\{ \frac{p}{r} \left(\frac{m}{p} \right)^{1/r} \|X\|_{L_r} : \max\{1, p/m\} \leq r \leq p \right\}.$$

Definition 5.10 If X is a random variable, for every $p \geq 1$ set

$$\|X\|_{(p)} = \sup_{1 \leq q \leq p} \frac{\|X\|_{L_q}}{\sqrt{q}}.$$

The (p) -norms are a local version of the ψ_2 norm, and clearly $\|X\|_{(p)} \lesssim \|X\|_{\psi_2}$. Using those norms one may obtain a more compact expression for the required moments.

Lemma 5.11 *There exist an absolute constant c such that for every $h \in H$, every $s > s_0$ and every $u > 0$,*

$$\|Z_s(h)\|_{L_{2u\eta_{s+1}}} \leq c\sqrt{u}\eta_{s+1}^{1/2} \|\Delta_s h\|_{(2u\eta_{s+1})}$$

and

$$\|Z_{s_0}(h)\|_{L_{2u\eta_{s_0+1}}} \leq c\sqrt{u}\eta_{s_0+1}^{1/2} \|\pi_{s_0} h\|_{(2u\eta_{s_0+1})}.$$

Proof. Let $Y_i = h(X_i)$ and observe that for every m , $\|(\sum_{i=1}^m Y_i^2)^{1/2}\|_{L_p} = \|\sum_{i=1}^m Y_i^2\|_{L_{p/2}}^{1/2}$. Since $m = u\ell_{s+1}$ and $p = 2u\eta_{s+1}$ then $p/2 \geq m$. Also, for every $r \leq p$, $\|Y\|_{L_r} \leq \sqrt{r}\|Y\|_{(p)}$, and applying Theorem 5.9,

$$\left\| \sum_{i=1}^m Y_i^2 \right\|_{L_{p/2}} \lesssim \|Y^2\|_{(p/2)} \sup_{2 \leq r \leq p} \frac{p}{\sqrt{r}} \left(\frac{m}{p} \right)^{1/r} \lesssim \|Y^2\|_{(p/2)} \frac{p}{\sqrt{\log(p/m)}}.$$

Hence, for our choice of p and m ,

$$\left\| \sum_{i=1}^m Y_i^2 \right\|_{L_{p/2}}^{1/2} \lesssim \sqrt{u}\eta_{s+1}^{1/2} \|Y^2\|_{(u\eta_{s+1})}^{1/2} = \sqrt{u}\eta_{s+1}^{1/2} \|Y\|_{(2u\eta_{s+1})}.$$

■

Corollary 5.12 *There exist absolute constants c_1 and c_2 for which the following holds. If, for $s > s_0$,*

$$\theta_{u,s}(\Delta_s h) = c_1 \sqrt{u}\eta_{s+1}^{1/2} \|\Delta_s h\|_{(2u\eta_{s+1})}$$

and

$$\theta_{u,s_0}(\pi_{s_0} h) = c_1 \sqrt{u}\eta_{s_0+1}^{1/2} \|\pi_{s_0} h\|_{(2u\eta_{s_0+1})},$$

then $Pr(\Omega_{1,u}) \geq 1 - 2\exp(-c_2 u\eta_{s_0})$.

Next, assume that $s_0 > 0$, and thus one has to bound $Pr(\Omega_{3,u})$.

Lemma 5.13 *There exists absolute constants c_1 , c_2 and c_3 such that, for every $u \geq c_1$, with probability at least $1 - 2\exp(-c_2 u \log N)$, for every $0 \leq s < s_0$ and every $h \in H$,*

$$\left(\sum_{i=1}^j ((\Delta_s h)^2(X_i))^* \right)^{1/2} \leq c_3 u \|\Delta_s h\|_{\psi_1} \sqrt{j} \log(eN/j) \sim u \|\Delta_s h\|_{\psi_1} \phi_1(j),$$

and a similar bound holds for $\pi_{s_0} h$.

Proof. Recall that for a fixed $\varepsilon > 0$ and every i , $\Pr(Y_i^* \geq y_i) \leq \exp(-\varepsilon i \log(eN/i))$. Let $\varepsilon \sim u \geq 1$ and observe that if $Y \in L_{\psi_1}$, then $y_i \lesssim u \|Y\|_{\psi_1} \log(eN/i)$ and

$$\Pr(\exists i \leq N : Y_i^* \geq y_i) \leq \exp(-c_1 u \log N). \quad (5.6)$$

Since the cardinality of the set $\cup_{s < s_0} \Delta_s H$ is at most $\sum_{s < s_0} 2^{\eta_{s+1}} \lesssim N^{c_2}$, (5.6) holds uniformly with probability at least $1 - \exp(-c_3 u \log N)$ for $u \geq c_4$. Therefore, on that event, for every $0 \leq s < s_0$ and every j ,

$$\left(\sum_{i=1}^j ((\Delta_s h)^2(X_i))^* \right)^{1/2} \leq c_5 u \|\Delta_s h\|_{\psi_1} \sqrt{j} \log(eN/j).$$

An identical argument holds for $\left(\sum_{i=1}^j ((\pi_{s_0} h)^2(X_i))^* \right)^{1/2}$. ■

Therefore, the event $\Omega_{1,u} \cup \Omega_{2,u} \cup \Omega_{3,u}$ has high probability, leading to the following decomposition result.

Corollary 5.14 *There exist absolute constants c_1 and c_2 for which the following holds. For every $u \geq c_1$, with probability at least $1 - 2\exp(-c_2 u \log N)$, for every $h \in H$ and every $I \subset \{1, \dots, N\}$,*

1. if $\eta_s \leq N$,

$$\left(\sum_{i \in I} ((\Delta_s h)^2(X_i))^* \right)^{1/2} \lesssim \sqrt{u} \eta_{s+1}^{1/2} \|\Delta_s h\|_{(2u\eta_{s+1})} + u \|\Delta_s h\|_{\psi_1} \phi_1(|I|),$$

and if $\eta_s > N$ then

$$\left(\sum_{i \in I} ((\Delta_s h)^2(X_i))^* \right)^{1/2} \lesssim \sqrt{u} \eta_{s+1}^{1/2} \|\Delta_s h\|_{(2u\eta_{s+1})}.$$

2.

$$\left(\sum_{i \in I} (h^2(X_i))^* \right)^{1/2} \lesssim \sqrt{u} \sum_{s > s_0} \eta_{s+1}^{1/2} \|\Delta_s h\|_{(2u\eta_{s+1}) + ud_{\psi_1} \phi_1(|I|) + R_{s_0}(h)},$$

where $R_{s_0}(h) \lesssim \sqrt{u} \eta_1 \|\pi_0 h\|_{(2u\eta_1)}$ if $s_0 = 0$ and $R_{s_0}(h) \lesssim ud_{\psi_1} \phi_1(|I|)$ otherwise.

Remark 5.15 Note that $\|\Delta_s h\|_{(2u\eta_{s+1})} \lesssim \|\Delta_s h\|_{\psi_2}$, and thus one may take $\theta_{u,s} \sim \sqrt{u} \eta_{s+1}^{1/2} \|\Delta_s h\|_{\psi_2}$. If $\eta_0 = 0$ and $\eta_s = 2^s$ for $s \geq 1$, then for an almost optimal admissible sequence,

$$\sum_{s > s_0} \eta_{s+1}^{1/2} \|\Delta_s h\|_{(2u\eta_{s+1})} \lesssim \gamma_2(H, \psi_2).$$

Although this estimate leads to an alternative proof of Theorem 4.1, it is not sharp enough to prove Theorem B, as the latter requires more accurate bounds on $\|\Delta_s h\|_{(2u\eta_{s+1})}$.

From here on we will assume that $\eta_0 = 0$ and that $\eta_s = 2^s$ for $s \geq 1$. If $s \geq s_0 \sim \log N$, set $\theta_{u,s}(\Delta_s t) = \sqrt{u} \eta_{s+1}^{1/2} \|\Delta_s h\|_{(2u\eta_{s+1})}$.

Theorem 5.16 There exist absolute constants c_1 and c_2 for which the following holds. If μ is an isotropic, unconditional log-concave measure, $H_T = \{\langle t, \cdot \rangle : t \in T\}$ and $(T_s)_{s \geq 0}$ is an admissible sequence of T , then for every $u \geq c_1$,

$$\theta_{u,s}(\langle \Delta_s t, \cdot \rangle) \leq c_2 u \left(2^s \|\Delta_s t\|_{\ell_\infty^n} + 2^{s/2} \|\Delta_s t\|_{\ell_2^n} \right).$$

Proof. Let $T \subset \mathbb{R}^n$, and identify it with the class of linear functionals $H_T = \{\langle t, \cdot \rangle : t \in T\}$ on (\mathbb{R}^n, μ) . By Borell's inequality [8], the ψ_1 and L_2 norms are c_1 -equivalent on \mathbb{R}^n , where c_1 is an absolute constant, and since μ is isotropic, then $\|\langle t, \cdot \rangle\|_{L_2} = \|t\|_{\ell_2^n}$. Moreover, there is an absolute constant c_2 such that for every $p \geq q$ and $t \in \mathbb{R}^n$,

$$\|\langle t, \cdot \rangle\|_{L_p} \leq c_2 \frac{p}{q} \|\langle t, \cdot \rangle\|_{L_q}.$$

Hence, for every $t \in \mathbb{R}^n$ and every $r \geq 1$,

$$\begin{aligned} \|\langle t, \cdot \rangle\|_{(rq)} &\leq \sup_{q \leq \ell \leq rq} \frac{\|\langle t, \cdot \rangle\|_{L_\ell}}{\sqrt{\ell}} + \|\langle t, \cdot \rangle\|_{(q)} \leq c_2 \sup_{q \leq \ell \leq rq} \frac{\ell}{q} \frac{\|\langle t, \cdot \rangle\|_{L_q}}{\sqrt{\ell}} + \|\langle t, \cdot \rangle\|_{(q)} \\ &\leq (c_2 \sqrt{r} + 1) \|\langle t, \cdot \rangle\|_{(q)}. \end{aligned}$$

Also, for any $t \in \mathbb{R}^n$ and any $p \geq n$, $\|\langle t, \cdot \rangle\|_{(p)} \leq 2c_2 \sqrt{\frac{p}{n}} \|\langle t, \cdot \rangle\|_{(n)}$. Therefore, if $u \geq 1$ and $\eta_s = 2^s \leq n$ then

$$\sqrt{u} \eta_{s+1}^{1/2} \|\langle \Delta_s t, \cdot \rangle\|_{(2u\eta_{s+1})} \lesssim u 2^{s/2} \|\langle \Delta_s t, \cdot \rangle\|_{(2^s)},$$

and if $2^s > n$ then

$$\sqrt{u} \eta_{s+1}^{1/2} \|\langle \Delta_s t, \cdot \rangle\|_{(2u\eta_{s+1})} \lesssim \sqrt{u} \frac{2^s}{n} \|\langle \Delta_s t, \cdot \rangle\|_{(n)}.$$

Note ([6] or [25], Proposition 3.4) that there is an isotropic convex body K such that for every $t \in \mathbb{R}^n$ and any $1 \leq p \leq n$, $\|\langle t, \cdot \rangle\|_{L_p(\mu)} \leq c_3 \|\langle t, \cdot \rangle\|_{L_p(K)}$. Moreover, since μ is unconditional, K is also unconditional and using the Bobkov-Nazarov Theorem [7] we get that

$$\|\langle t, \cdot \rangle\|_{L_p(\mu)} \leq c_3 \|\langle t, \cdot \rangle\|_{L_p(K)} \leq c_4 \|\langle t, \cdot \rangle\|_{L_p(K_1)},$$

where K_1 is an isotropic image of B_1^n .

The moments of every linear functional $\langle t, \cdot \rangle$ relative to the volume measure of an isotropic position of B_1^n are well known [15]: namely, for $1 \leq p \leq n$,

$$\|\langle t, \cdot \rangle\|_{L_p(K_1)} \sim p \|t\|_\infty + \sqrt{p} \left(\sum_{i=p+1}^n (t_i^2)^* \right)^{1/2}.$$

Combining the two estimates, for $p \leq n$ and any $t \in \mathbb{R}^n$,

$$\begin{aligned} \|\langle t, \cdot \rangle\|_{(p)} &\leq \sup_{q \leq p} \frac{\|\langle t, \cdot \rangle\|_{L_q(K_1)}}{\sqrt{q}} \sim \sup_{q \leq p} \left(\sqrt{q} \|t\|_\infty + \left(\sum_{i=q+1}^n (t_i^2)^* \right)^{1/2} \right) \\ &\leq \sqrt{p} \|t\|_{\ell_\infty^n} + \|t\|_{\ell_2^n}. \end{aligned}$$

Thus, for $2^s \leq n$,

$$2^{s/2} \|\langle \Delta_s t, \cdot \rangle\|_{(2^s)} \leq 2^s \|\Delta_s t\|_{\ell_\infty^n} + 2^{s/2} \|\Delta_s t\|_{\ell_2^n},$$

and if $2^s > n$,

$$\frac{2^s}{\sqrt{n}} \|\langle \Delta_s t, \cdot \rangle\|_{(n)} \leq 2^s \|\Delta_s t\|_{\ell_\infty^n}.$$

■

Note that for an almost optimal admissible sequence,

$$\sum_{s \geq 0} 2^s \|\Delta_s t\|_{\ell_\infty^n} + 2^{s/2} \|\Delta_s t\|_{\ell_2^n} \lesssim \gamma_1(T, \ell_\infty^n) + \gamma_2(T, \ell_2^n).$$

It turns out that $\gamma_1(T, \ell_\infty^n) + \gamma_2(T, \ell_2^n)$ can be completely characterized by the following beautiful result due to Talagrand [31, 32].

Theorem 5.17 *There exist absolute constants c and C for which the following holds. Let $(y_i)_{i=1}^n$ be independent, standard exponential variables. Then, for every $T \subset \mathbb{R}^n$,*

$$c \mathbb{E} \sup_{t \in T} \sum_{i=1}^n y_i t_i \leq \gamma_1(T, \ell_\infty) + \gamma_2(T, \ell_2) \leq C \mathbb{E} \sup_{t \in T} \sum_{i=1}^n y_i t_i.$$

Recall that if $(y_i)_{i=1}^n$ are standard exponential random variables and $T \subset \mathbb{R}^n$, then we denote $E(T) = \mathbb{E} \sup_{t \in T} \sum_{i=1}^n y_i t_i$ and $d_2(T) = \sup_{t \in T} \|t\|_2$.

Combining the estimates above, it follows that on $\Omega_{1,u} \cap \Omega_{2,u} \cap \Omega_{3,u}$, $P_\sigma T$ satisfies Definition 3.1 with $\theta_s = 2^s \|\cdot\|_{\ell_\infty^n} + 2^{s/2} \|\cdot\|_{\ell_2^n}$ for $s \geq s_0$ and $\theta_s = 0$ otherwise, $\gamma \lesssim E(T)$, $\phi \sim \phi_1$, $\|(\langle X_i, t \rangle)_{i=1}^N\| = \|\langle t, \cdot \rangle\|_{\psi_1} \sim \|t\|_{\ell_2^n}$ and $\alpha \sim u$. Therefore, $B_4 \lesssim \gamma_2(T, \ell_2) \lesssim E(T)$.

Theorem 5.18 *There exist absolute constants c_1, c_2, c_3 and c_4 for which the following holds. For every $u \geq c_1$, With μ^N -probability at least $1 - 2 \exp(-c_2 u \log N)$, the set $V = P_\sigma T$ satisfies that*

$$\sup_{v \in V} \left| \sum_{i=1}^N \varepsilon_i v_i^2 \right| \leq c_3 r u^2 \left(d_2(T) \sqrt{N} E(T) + (E(T))^2 \right)$$

with probability at least $1 - 2 \exp(-c_4 r^2)$ with respect to the Bernoulli random variables.

5.3 Proofs of Theorems A and B

The final step we need for the proofs of Theorem A and Theorem B is a version of the Ginè-Zinn symmetrization Theorem (see, e.g. [14, 33]), which enables one to pass from the Bernoulli process indexed by random coordinate projections of a class of functions, to the empirical process indexed by the class.

Theorem 5.19 *Let F be a class of functions and for every $x > 0$, set $\beta_N(x) = \inf_{f \in F} \Pr(|\sum_{i=1}^N f(X_i) - \mathbb{E}f| > x/2)$. Then*

$$\beta_N(x) \Pr_X \left(\sup_{f \in F} \left| \sum_{i=1}^N f(X_i) - \mathbb{E}f \right| > x \right) \leq 2 \Pr_{X \otimes \varepsilon} \left(\sup_{f \in F} \left| \sum_{i=1}^N \varepsilon_i f(X_i) \right| > x/4 \right).$$

To apply Theorem 5.19, one has to identify the right value x for which $\beta_N(x) \geq 1/2$. In our case, $F = H^2$, and thus one has to show that if x is large enough, then $\sup_{h \in H} \Pr(|\sum_{i=1}^N h^2(X_i) - \mathbb{E}h^2| > x/2) \leq 1/2$.

Lemma 5.20 *Let H be a class of functions which is bounded in L_q and consider the empirical process indexed by $F = \{h^2 : h \in H\}$. If $q \geq 4$ and $x \gtrsim d_{L_q}^2 \sqrt{N}$ then $\beta_N(x) \geq 1/2$ and the same holds if $2 < q < 4$ and $x \gtrsim_q d_{L_q}^2 N^{2/q}$.*

Proof. The first part of the claim follows from an application of Chebyshev's inequality, and is omitted. For the second part, fix $r > 0$, set $V = (h^2(X_i))_{i=1}^N$, and since $\Pr(|h^2(X)| \geq d_{L_q}^2 (rN/i)^{2/q}) \leq i/(rN)$ then

$$\Pr \left(V_i^* \geq d_{L_q}^2 (rN/i)^{2/q} \right) \lesssim \binom{N}{i} \cdot (i/rN)^i \leq \exp(-i \log(er)) = (er)^{-i}. \quad (5.7)$$

Moreover, for $r_0 \sim c_2$, $\Pr(\exists i : |h^2(X_i)| \geq r_0 d_{L_q}^2 N^{2/q}) \leq 1/10$. Hence, a truncation argument shows that without loss of generality we may assume that $\|h^2\|_{L_\infty} \leq r_0 d_{L_q}^2 N^{2/q}$. Applying the L_∞ estimate for the largest two coordinates of V and (5.7) for the rest, it follows that

$$\Pr(\|V\|_{\ell_2^N} \geq c_q(r_0 + r) d_{L_q}^2 N^{2/q}) \lesssim \sum_{i=3}^N r^{-i} \lesssim r^{-2}.$$

Hence, under the truncation assumption,

$$\begin{aligned} \mathbb{E} \left| \sum_{i=1}^N h^2(X_i) - \mathbb{E}h^2 \right| &\lesssim \mathbb{E}_X \mathbb{E}_\varepsilon \left| \sum_{i=1}^N \varepsilon_i h^2(X_i) \right| \lesssim \mathbb{E}_X \left(\sum_{i=1}^N h^4(X_i) \right)^{1/2} = \mathbb{E} \|V\|_{\ell_2^N} \\ &\lesssim_q r_0 d_{L_q}^2 N^{2/q}, \end{aligned}$$

showing that it suffices to take $x \sim_q d_{L_q}^2 N^{2/q}$ as claimed. \blacksquare

Since x/N is well within our range, one may complete the proofs of Theorem A and Theorem B.

Proof of Theorem A. For $q > 4$, let $\rho_{r,u} \sim_{\kappa_1, \kappa_2, q} ru(\sqrt{n/N} + n/N)$ for $u \gtrsim_{\kappa_1, \kappa_2, q} c_1$, and $r \geq c_2$. If $2 < q < 4$ set $\rho_{r,u} \sim_{\kappa_1, \kappa_2, q} ru(n/N)^{1-2/q}$ for $u \gtrsim_{\kappa_1, \kappa_2, q} \log(eN/n)$ and $r \geq c_3$. Then,

$$\begin{aligned}
& Pr_X \left(\sup_{t \in S^{n-1}} \left| \frac{1}{N} \sum_{i=1}^N \langle t, X_i \rangle^2 - \mathbb{E} \langle t, X \rangle^2 \right| \geq \rho_{r,u} \right) \\
& \leq 4\mathbb{E}_X Pr_\varepsilon \left(\left| \frac{1}{N} \sum_{i=1}^N \varepsilon_i \langle t, X_i \rangle^2 \right| \geq \rho_{r,u}/4 \right) \\
& \lesssim Pr_X((\Omega_{1,u} \cap \Omega_{2,u} \cap \Omega_{3,u})^c) + Pr_\varepsilon \left(\left| \frac{1}{N} \sum_{i=1}^N \varepsilon_i \langle t, X_i \rangle^2 \right| > \rho_{r,u}/4 \middle| \Omega_{1,u} \cap \Omega_{2,u} \cap \Omega_{3,u} \right) \\
& \lesssim \exp(-c_4 n).
\end{aligned}$$

■

Proof of the quantitative Bai-Yin Theorem.

To prove the quantitative version of the Bai-Yin Theorem one has to combine Theorem A with a conditioning argument. Consider the vector $X = (\xi_1, \dots, \xi_n)$ with $\xi \in L_q$ for some $q > 4$, and let ν be the measure on \mathbb{R}^n given by $\nu = X|cn^{1/p}B_p^n$; that is, ν is given by the conditioning of X to the unconditional body $cn^{1/p}B_p^n$ for a suitable choice of c and p . Clearly, ν is unconditional and satisfies the p -small diameter L_q moment assumption, and thus, falls within the realm of Theorem A. Therefore, if the event $\mathcal{A} = \{\max_{i \leq N} \|X_i\|_{\ell_p^n} \leq cn^{1/p}\}$ has high enough probability, the quantitative version of the Bai-Yin Theorem follows from Theorem A, because for every event \mathcal{B} ,

$$Pr((X_i)_{i=1}^N \in \mathcal{B}) \leq Pr((X_i)_{i=1}^N \in \mathcal{B} | X_1, \dots, X_N \in cn^{1/p}B_p^n)Pr(\mathcal{A}) + Pr(\mathcal{A}^c).$$

Hence, the final step in the proof of our version of the Bai-Yin Theorem is to show that if $\xi \in L_q$ for $q > 2$, there is some $p > 2$ for which \mathcal{A} has a large measure.

Recall that for every $v \in \mathbb{R}^n$, $\|v\|_{\ell_{p,\infty}^n} = \max_{i \leq n} v_k^*/k^{1/p}$, and since $\ell_r^n \subset \ell_{p,\infty}^n \subset \ell_p^n$ for every $r < p$, it suffices to show that $\max_{i \leq N} \|X\|_{\ell_{p,\infty}^n} \lesssim n^{1/p}$ for some $p > 2$ with high enough probability.

Lemma 5.21 *For every $q > 4$ and $2 < p < q$, there exist constants c_1 and c_2 that depend on q and p for which the following holds. If $\xi \in L_q$,*

$X = (\xi_1, \dots, \xi_n)$ and X_1, \dots, X_N are independent copies of X , then

$$Pr(\max_{1 \leq i \leq N} \|X_i\|_{\ell_{p,\infty}^n} \geq c_1 \|\xi\|_{L_q} n^{1/p}) \leq \frac{c_2 N}{n^{\frac{q}{p}-1}}.$$

Proof. If $A = \|\xi\|_{L_q}$ then $Pr(|\xi| \geq At) \leq t^{-q}$, and for every $1 \leq k \leq n$, $Pr(\xi_k^* \geq t) \leq \binom{n}{k} (Pr(|\xi| \geq t))^k$. Therefore, if $p < q$ and $y > e$ then

$$\begin{aligned} Pr(\xi_k^* \geq A(ny/k)^{1/p}) &\leq \exp(k \log(en/k) - k(q/p) \log(ny/k)) \\ &\leq \exp(-k(\frac{q}{p} - 1) \log(ny/k)). \end{aligned}$$

Using this estimate for every $k = 2^j$ and summing the probabilities, it follows that for every q and p there is a constant $c_{q,p}$ for which $\|X\|_{\ell_{p,\infty}^n} \lesssim n^{1/p}$ with probability at least $1 - c_{q,p} n^{1-q/p}$, and in particular, $Pr(\max_{i \leq N} \|X_i\|_{\ell_{p,\infty}^n} \geq cn^{1/p}) \leq c_{q,p} N/n^{(q/p)-1}$, as claimed. ■

Combining Lemma 5.21 with Theorem A concludes the proof of the quantitative Bai-Yin Theorem. ■

Proof of Theorem B. If $r \sim u$, with probability at least $1 - 2 \exp(-c_3 u^2)$ with respect to the Bernoulli random variables,

$$\sup_{v \in P_{\sigma} T} \left| \frac{1}{N} \sum_{i=1}^N \varepsilon_i v_i^2 \right| \lesssim_{u^3} d_2(T) \frac{E(T)}{\sqrt{N}} + \frac{E^2(T)}{N}.$$

Since $d_2(T)E(T)/\sqrt{N}$ is a “legal” choice in the Giné-Zinn symmetrization theorem, the proof is concluded. ■

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